Phenomenology and theory of neutrino mixing and oscillations

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Bruno M. Pontecorvo was born in 1913 in Pisa (Italy) in a wealthy family. His father was owner of a textile factory, mother was from family of a doctor. There were eight children in the family: five brothers and three sisters. All were talented. Three became famous: biologist Guido, film director Gillo, physicist Bruno.

Afer the school Bruno entered the Engineer Faculty of the Pisa University

After 2 years of the Engineer Faculty he decided to switch to physics

His oldest brother Guido recommended him to go to Rome, where E.Fermi and his group worked

Bruno passed an exam (Fermi, Rasetti), and was accepted to the Rome University. He became Fermi student (1932)

B.P. plaid an important role in the discovery of the effect of slow neutrons (1934) All practical applications of neutrons are based on this effect)

In 1936-40 Bruno worked in Paris in Jollot Curie group (nuclear isomerism)

In 1940-42 USA. B.P. invented and applied a new method of the searching for oil (neutron well logging)

In 1943-49 Canada. Scientific leader of the first research reactor in Canada, first experiments on the study of μ-decay, first experiment on the measurement of neutrino mass, first method of neutrino detection (Cl-Ar radiochemical method), idea of $\mu - e$ universality of the weak interactions,...

In 1950-93 Dubna, JINR. First experiments on Dubna synchrocyclotron, first proposal of accelerator neutrino experiments (the experiment was done at BNL and allowed to establish existence of the muon neutrino (ν_μ)), first idea of neutrino oscillations, development of this idea,...

B. M. Pontecorvo was great neutrino physicist, one of the creators of modern neutrino physics. He was extremely charming, intelligent and gifted person. Physics for him was the most important. But he also liked very much tennis, literature, music, underwater fishing, ...

Observation of neutrino oscillations in solar, atmospheric, reactor and accelerator neutrino experiments is one of the most important recent discovery in particle physics. It is a common opinion that with the discovery of neutrino oscillation a new, beyond the SM physics was unveiled

The observation of neutrino oscillations means that

- Neutrinos have small but different from zero masses.
- Fields of neutrinos with definite masses enter into CC current in the mixed form

All existing weak interaction data can be described by the standard CC and NC Hamiltonians

The Standard CC lepton interaction

$$\mathcal{L}_{\mathcal{I}}^{\mathcal{CC}}(x) = -\frac{g}{2\sqrt{2}} j_{\alpha}^{CC}(x) W^{\alpha}(x) + \mathrm{h.c.}$$

$$j_{\alpha}^{CC}(x) = 2 \sum_{l=e,\mu,\tau} \bar{\nu}_{lL}(x)\, \gamma_{\alpha}\, l_L(x)$$

From neutrino oscillation experiments follow that $\nu_{lL}(x)$ is "mixed field"

$$\nu_{lL}(x) = \sum_{i=1}^{3} U_{li}\, \nu_{iL}(x)$$

$\nu_i(x)$ is the field of neutrino with mass m_i

U is 3×3 unitary Pontecorvo-MNS mixing matrix

The most important consequences of the neutrino mixing are neutrino oscillations - periodical transitions between different flavor neutrinos ($\nu_l \rightleftarrows \nu_{l'}$) in neutrino beams

The standard probability of the transition $\nu_l \to \nu_{l'}$ in vacuum has the form

$$\mathrm{P}(\nu_l \to \nu_{l'}) = |\sum_{i=1}^{3} U_{l'i}\, e^{-i\frac{\Delta m^2_{2i} L}{2E}}\, U^*_{li}|^2$$

$\Delta m^2_{ik} = m^2_k - m^2_i$, L is the source-detector distance, E is the neutrino energy

In the expression for $\mathrm{P}(\nu_l \to \nu_{l'})$ the relative phase $\frac{\Delta m^2_{2i} L}{2E}$ comes from propagation of neutrinos with definite masses, factors U^*_{li} and U_{li} comes from states of initial and final flavor neutrinos

Coherent sum over the states of neutrinos with definite masses is performed.

If we take into account the unitarity of the PMNS matrix we can present the transition probability in another form

$$\mathrm{P}(\nu_l \to \nu_{l'}) = |\delta_{l'l} + \sum_{i=1,3} U_{l'i}\,(e^{-i\frac{\Delta m^2_{2i} L}{2E}} - 1)\, U^*_{li}|^2$$

The unitary PMNS mixing matrix is characterized by three mixing angles and one CP phase. It can be obtained by three Euler rotations

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Transition probability depends on six parameters: Δm^2_{12} and Δm^2_{23}, three mixing angles θ_{12}, θ_{23}, θ_{13} and CP-phase δ

From experimental data follow that two parameters are small

$$\frac{\Delta m^2_{12}}{\Delta m^2_{23}} \simeq \frac{1}{30}$$

$$\sin^2\theta_{13} \lesssim 5 \cdot 10^{-2}$$

In the leading approximation a rather simple picture of neutrino oscillations is emerged

Let us consider neutrino oscillations under the assumptions that

$\sin^2\theta_{13} = 0$ and $\Delta m^2_{12} \ll \Delta m^2_{23}$

In the atmospheric region of $\frac{L}{E}$ ($\frac{\Delta m^2_{23} L}{2E} \gtrsim 1$) we have

$$\mathrm{P}(\nu_\mu \to \nu_\tau) \simeq |U_{\tau 3}\,(e^{-i\frac{\Delta m^2_{23} L}{2E}} - 1)\,U^*_{\mu 3}|^2 = \frac{1}{2}\sin^2 2\theta_{23}\,(1 - \cos\Delta m^2_{23}\frac{L}{2E})$$

$$\mathrm{P}(\nu_\mu \to \nu_e) \simeq 0$$

For the $\nu_\mu \to \nu_\mu$ survival probability we find

$$P(\nu_\mu \to \nu_\mu) = 1 - \sum_{l'=\tau,e} P(\nu_\mu \to \nu_{l'}) \simeq 1 - \frac{1}{2}\sin^2 2\theta_{23}\,(1 - \cos\Delta m^2_{23}\frac{L}{2E})$$

In the reactor KamLAND region ($\frac{\Delta m^2_{12} L}{2E} \gtrsim 1$) we have $\frac{\Delta m^2_{23} L}{2E} \gg 1$. The contribution of the "large" Δm^2_{23} disappears due to averaging over neutrino spectrum etc.

For the $\nu_e \to \nu_e$ survival probability we have

$$P(\bar\nu_e \to \bar\nu_e) = 1 - \frac{1}{2}\sin^2 2\theta_{12}\,(1 - \cos\Delta m^2_{12}\frac{L}{2E})$$

The probability of the solar neutrinos to survive is given by the two-neutrino ν_e survival probability in matter which depends on Δm^2_{12}, $\sin^2\theta_{12}$ and electron number density

In analysis of neutrino oscillations data this first approximation dominates : four parameters (Δm^2_{12}, Δm^2_{23}, $\sin^2\theta_{23}$, $\sin^2\theta_{12}$) can be determined from the data. For other parameters only upper bounds can be inferred

From analysis of the Super-K atmospheric neutrino data

$$\Delta m^2_{23} = 2.19^{+0.14}_{-0.13}\ \mathrm{eV}^2, \quad \sin^2 2\theta_{23} > 0.96$$

$$\sin^2\theta_{13} < 7\cdot 10^{-2}\ (\textit{normal spectrum})$$

$$\sin^2\theta_{13} < 1.3\cdot 10^{-1}\ (\textit{inverted spectrum})$$

The Super-K evidence for neutrino oscillations was confirmed by the accelerator long-baseline K2K and MINOS experiments

From two-neutrino analysis of the MINOS data

$$\Delta m^2_{23} = (2.43\pm 0.13)\cdot 10^{-3}\ \mathrm{eV}^2, \quad \sin^2 2\theta_{23} > 0.90$$

From global analysis of the solar and reactor KamLAND data

$$\Delta m_{12}^2 = (7.59^{+0.21}_{-0.21}) \cdot 10^{-5}\ \mathrm{eV}^2, \quad \sin^2\theta_{12} = 0.31^{+0.02}_{-0.02}$$

$$\sin^2\theta_{13} < 6 \cdot 10^{-2}$$

From the reactor CHOOZ data

$$\sin^2\theta_{13} < 4 \cdot 10^{-2}$$

Notice that for the absolute value of neutrino mass from Mainz and Troitsk tritium data the following bound was obtained

$$m_\beta \leq 2.2\ \mathrm{eV}$$

These experiments are much less sensitive to neutrino mass than oscillation experiments. This is connected with the interference nature of neutrino oscillations

Neutrino masses and neutrino mixing are due to NEUTRINO MASS TERM of the Lagrangian

Generation of mass terms is the most difficult part of the modern theory

We believe that mass terms of quarks and leptons are due to spontaneous violation of the EW symmetry. Origin of the neutrino mass term is unknown at present

We will consider all possible neutrino mass terms

Neutrino mass term is a sum of Lorenz-invariant products of the left-handed and right-handed components of neutrino fields

$\nu_{lL}(x)$ ($l = e, \mu, \tau$), components of $SU(2)$ doublets, enter into interaction ("active components")

$\nu_{lR}(x)$ are $SU(2)$ singlets; do not enter into interaction ("sterile components")

In the neutrino mass term active and sterile components could enter

Special case. DIRAC mass term

$$\mathcal{L}^{\mathrm{D}}(x) = -\bar{\nu}_L(x)\, M^{\mathrm{D}}\, \nu_R(x) + \mathrm{h.c.}$$

$$\nu_L = \begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix}, \quad \nu_R = \begin{pmatrix} \nu_{eR} \\ \nu_{\mu R} \\ \nu_{\tau R} \end{pmatrix}$$

M^{D} is a 3×3 complex matrix

After the standard diagonalization ($M = U^{\dagger} m V$, U and V are unitary matrices, $m_{ik} = m_i \delta_{ik}$

$$\mathcal{L}^{\mathrm{D}}(x) = \sum_{i=1}^{3} m_i\, \bar{\nu}_i(x)\, \nu_i(x)$$

$\nu_i(x)$ is the field of neutrino with the mass m_i

$$\nu_{lL}(x) = \sum_{i=1}^{3} U_{li}\ \nu_{iL}(x)$$

The Dirac mass term is invariant under the global phase transformations

$$\nu_i(x) \to e^{i\,\Lambda}\nu_i(x),\;\; l(x) \to e^{i\,\Lambda}\, l_i(x),\;\; q(x) \to q(x)$$

Λ is an arbitrary constant

The total lepton number $L = L_e + L_\mu + L_\tau$ is conserved

$\nu_i(x)$ is the four-component Dirac field of neutrinos and antineutrinos with the same mass m_i and different lepton numbers

$$L(\nu_i) = 1, L(\bar{\nu}_i) = -1$$

(for example, neutrinoless double β-decay

$(A, Z) \to (A, Z-2) + e + e$ is forbidden)

Remark

$\nu(x)$ and $\nu^c(x) = C\bar{\nu}^T(x)$ $(C\gamma_\alpha^T C^{-1} = -\gamma_\alpha,\;\; C^T = -C)$ neutrino field and conjugated neutrino field

$\nu_{L,R}(x)$ is left(right) component

$(\nu_{L,R}(x))^c$ is right(left) component

The most general DIRAC AND MAJORANA MASS TERM

$$\mathcal{L}^{\mathrm{D+M}} = -\frac{1}{2}\,\bar{\nu}_L\,M_L(\nu_L)^c - \bar{\nu}_L\,M^{\mathrm{D}}\,\nu_R - \frac{1}{2}\,\overline{(\nu_R)^c}\,M_R\nu_R + \mathrm{h.c.}$$

M_L and M_R are symmetrical 3×3 matrices

After the standard diagonalization of the mass term we find

$$\mathcal{L}^{\mathrm{D+M}}(x) = -\frac{1}{2}\sum_{i=1}^{6} m_i\,\bar{\nu}_i(x)\,\nu_i(x)$$

$\nu_i(x)$ is the field of neutrino with mass m_i which satisfies the Majorana condition

$$\nu_i(x) = \nu_i^c(x) = C\bar{\nu}_i^T(x)$$

Majorana field

$$\nu(x) = \int N_p[a_r(p)e^{-ipx}u^r(p) + a_r^\dagger(p)e^{ipx}u^r(-p)]d^3p$$

$a_r(p)$ ($a_r^\dagger(p)$) is the operator of absorption (creation) of neutrino

No antineutrinos. $(\nu_i \equiv \bar{\nu}_i)$

No global gauge invariance $\rightarrow$ no conserved lepton number $\rightarrow$ no way to distinguish neutrino and antineutrino

Mixing relations

$$\nu_{lL} = \sum_{i=1}^{6} U_{li}\,\nu_{iL}, \qquad (\nu_{lR})^c = \sum_{i=1}^{6} U_{\bar{l}i}\,\nu_{iL} \quad l = e, \mu, \tau$$

Active ν_{lL} and sterile $(\nu_{lR})^c$ fields are mixtures of the fields of six Majorana neutrinos with definite masses

Special case. Majorana mass term

$$\mathcal{L}^{\mathrm{M}} = -\frac{1}{2}\,\bar{\nu}_L\,M_L(\nu_L)^c + \mathrm{h.c.}$$

M_L is symmetrical 3×3 complex matrix

After the standard diagonalization

$$\mathcal{L}^{\mathrm{M}}(x) = -\frac{1}{2}\sum_{i=1}^{3} m_i \, \bar{\nu}_i(x) \, \nu_i(x)$$

$\nu_i(x) = \nu_i^c(x) = C\bar{\nu}_i^T(x)$ is Majorana field (i=1,2,3)

Mixing relation

$$\nu_{lL} = \sum_{i=1}^{3} U_{li} \, \nu_{iL} \quad l = e, \mu, \tau$$

The most economical case: left-handed neutrino fields enter into interaction and into mass term. No sterile neutrino fields in the Lagrangian

General conclusions from the consideration of possible neutrino mass terms

I. Neutrino with definite masses ν_i can be Dirac particle (neutrino and antineutrino differ by a conserved lepton number) or Majorana particle (neutrino and antineutrino are identical)

What is the nature of massive neutrinos? This is the most fundamental problem solution of which will be extremely important for the understanding of the origin of neutrino masses

The problem can be solved by the observation (or non observation) of $0\nu\beta\beta$-decay $(A, Z) \to (A, Z+2) + e + e$

If $0\nu\beta\beta$-decay will be observed ν_i are Majorana particles

If $0\nu\beta\beta$-decay will be not observed but neutrino mass will be measured in tritium experiments, ν_i are Dirac particles

II. Number of massive neutrinos can be larger than the number of flavor neutrinos

From the measurement of the width of the decay $Z \to \nu_l + \bar{\nu}_l$ (LEP)

$$N_{\nu_l} = 2.984 \pm 0.008$$

If the number of light ν_i is larger than three

$$\nu_{lL} = \sum_{i=1}^{3+n_s} U_{li}\, \nu_{iL} \quad l = e, \mu, \tau$$

$$\nu_{sL} = \sum_{i=1}^{3+n_s} U_{si}\, \nu_{iL} \quad s = s_1, ...s_{n_s}$$

All flavor neutrinos $(\nu_e, \nu_\mu, \nu_\tau)$ were observed in direct experiments

Sterile neutrinos ν_s can not be produced in weak processes

There are two ways to reveal existence of the sterile neutrinos

I. Sterile neutrinos can be produced in oscillations $\nu_l \to \nu_s$

If neutrinos are detected through NC processes transition probability is given

$$\sum_{l'=e,\mu.\tau} \mathrm{P}(\nu_l \to \nu_{l'}) = 1 - \sum_{s} \mathrm{P}(\nu_l \to \nu_s)$$

If there are no transitions into sterile neutrinos no oscillations

If there are transitions into sterile neutrinos transition probability depends on $\frac{L}{E}$ and oscillates

II. The number of neutrino mass-squared differences in the probability of the transition $\nu_l \to \nu_{l'}$ is equal $2 + n_s$

Thus, if the number of neutrino mass-squared differences is larger than two. sterile neutrinos exist

During many years exist indications in favor of $\bar{\nu}_\mu \to \bar{\nu}_e$ transitions with $\Delta m^2 \simeq 1\mathrm{eV}^2$ (LSND)

In the MiniBooNE experiment this indications was checked

In the channel $\nu_\mu \to \nu_e$ LSND result was not confirmed

In the channel $\bar{\nu}_\mu \to \bar{\nu}_e$ some indication in favor of the transition was found Further experiments are necessary, Why neutrino oscillations is a signature of a new physics?

Neutrino are massive and mixed, but quarks are also massive and mixed

In the quarks case we believe that quarks masses are due to SM physics

Why neutrinos...

Because neutrino masses are many order of magnitude smaller than masses of quarks and leptons

Let us consider for illustration the third family

$$m_t \simeq 1.7 \cdot 10^2 \ \mathrm{GeV}, \quad m_b \simeq 4.7 \ \mathrm{GeV}$$

$$m_3 \leq 2.2 \ 10^{-9} \ \mathrm{GeV}, \quad m_\tau \simeq 1.8 \ \mathrm{GeV}$$

Very unlikely that masses of quarks, leptons and neutrinos are of the same origin

We believe that masses of quarks and lepton are due to the SM Higgs mechanism

For neutrino masses a new (or additional) mechanism is needed

Special case of the D+M mass term. The seesaw mechanism

Let us consider D+M mass term in the simplest case of one generation

$$\mathcal{L}^{\mathrm{D+M}} = -\frac{1}{2}\, m_L \bar{\nu}_L (\nu_L)^c - m_D \bar{\nu}_L \nu_R - \frac{1}{2}\, m_R \overline{(\nu_R)^c} \nu_R + \mathrm{h.c.}$$

Assume that $m_{L,R}$ and m_D are real parameters

The mass term can be easily diagonalized

$$\mathcal{L}^{\mathrm{D+M}} = -\frac{1}{2} \sum_{i=1,2} m_i \, \bar{\nu}_i \, \nu_i$$

$\nu_{1,2}$ are Majorana fields and we have mixing relations

$$\nu_L = \cos\theta \ \nu_{1L} + \sin\theta \ \nu_{2L} \quad (\nu_R)^c = -\sin\theta \ \nu_{1L} + \cos\theta \ \nu_{2L}$$

Neutrino masses $m_{1,2}$ and mixing angle θ are connected with the parameters $m_{L,R}$ and m_R

$$m_{1,2} = \left| \frac{1}{2}(m_R + m_L) \mp \frac{1}{2}\sqrt{(m_R - m_L)^2 + 4\, m_D^2} \right|$$

$$\tan 2\,\theta = \frac{2m_D}{m_R - m_L}$$

We will assume now

1. There is no left-handed Majorana mass term $m_L = 0$
2. The Dirac mass term is generated by the Standard Higgs mechanism, i.e. m_D is of the order of a mass of quark or lepton
3. The right-handed Majorana mass term is the only term which does not conserve lepton number. We assume that the lepton number is violated at a scale which is much larger than the electroweak scale $m_R \equiv M_R \gg m_D$

The seesaw masses of the Majorana particles are given

$$m_1 \simeq \frac{m_D^2}{M_R} \ll m_D, \quad m_2 \simeq M_R \gg m_D$$

The mixing angle

$$\theta \simeq \frac{m_D}{M_R} \ll 1.$$

In the seesaw approach the smallness of neutrino masses is connected with violation of the total lepton number at a large scale given by M_R.The suppression factor is given by the ratio of the electroweak scale and the scale of the violation of the lepton number $(\frac{m_D}{M_R})$

If $m_D \simeq m_t \simeq 170$ GeV and $m_1 \simeq 5 \cdot 10^{-2}$ we find $M_R \simeq \frac{m_D^2}{m_1} \simeq 10^{15}$ GeV.

In the case of three families the seesaw matrix has the form

$$M = \begin{pmatrix} 0 & m_D \\ m_D^T & M_R \end{pmatrix}$$

m_D and M_R are 3×3 matrices and $M_R \gg m_D$.
The matrix M can be presented in block-diagonal form by the unitary transformation

$$U^T\, M\, U = \begin{pmatrix} -m_D M_R^{-1} m_D^T & 0 \\ 0 & M_R \end{pmatrix}$$

The 3×3 Majorana neutrino mass matrix is given by

$$m_\nu = -m_D\; M_R^{-1}\; m_D^T.$$

There are many parameters in the matrix, but large denominator ensure the smallness of neutrino masses with respect to masses of leptons and quarks

General conclusions from the seesaw mechanism

I. Neutrinos are Majorana particles.

II. Neutrino masses are much smaller than lepton and quark masses.

III.Heavy Majorana particles, the seesaw partners of neutrinos, must exist.

CP-violating decays of heavy Majorana particles in the early Universe is considered as a possible source of the barion asymmetry of the Universe (lepton asymmetry generate barion asymmetry)

We have discussed the standard seesaw mechanism

Equivalent more specific possibility of the explanation of the smallness of neutrino masses is based on the effective Lagrangian approach

Assume that the Lagrangian is the sum of the SM Lagrangian with massless neutrinos and non renormalizable effective Lagrangian

$$\mathcal{L}_{\rm eff} = -\sum_{l' l = e,\mu,\tau} (\psi^T_{l' L}\, C^{-1} \tau_2\, \phi)\,(y_{l' l}\frac{1}{M})(\phi^T\, \tau_2\, \psi_{lL}) + \mathrm{h.c.}$$

$$\psi_{lL} = \begin{pmatrix} \nu_{lL} \\ l_L \end{pmatrix} \qquad \phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$$

ψ_{lL} and ϕ are $SU(2)$ lepton and Higgs doublets

$\mathcal{L}_{\rm eff}$ is dimension-five operator which does not conserve L

Because the Lagrangian has dimension four, M has a dimension of a mass (assuming that $y_{l' l}$ are dimensionless coefficients)

The parameter M characterizes a (large) scale at which the Standard Model is violated

If we put

$$\phi(x) = \left(\begin{array}{c} 0 \\ \frac{v+H(x)}{\sqrt{2}}, \end{array} \right)$$

electroweak $SU(2) \times U(1)$ symmetry will be spontaneously broken

$v \simeq 246$ GeV is vacuum expectation value of the Higgs field

$H(x)$ is the field of Higgs boson

After the spontaneous violation of the symmetry the left-handed Majorana mass term is generated from the effective Lagrangian

$$\mathcal{L}^{\mathrm{M}} = \frac{1}{2} \sum_{l'l} \nu^{T}_{l'L} \; C^{-1} M^{M}_{l'l} \; \nu_{lL} + \mathrm{h.c.}$$

The matrix

$$M^{M}_{l'l} = \frac{y_{l'l} \; v^2}{M}$$

has the typical seesaw structure

The effective Lagrangian $\mathcal{L}_{\rm eff}$ can be induced by three different interactions between SM particles and very heavy Majorana or neutral scalar (beyond the SM) particles:

I.An interaction of lepton-Higgs pairs with a heavy Majorana singlet fermion N_R . The Lagrangian $\mathcal{L}_{\rm eff}$ is induced by the diagrams with exchange of a virtual N_R between lepton-Higgs pairs

II. An interaction of lepton pairs and Higgs pair with triplet heavy scalar boson Δ The effective Lagrangian $\mathcal{L}_{\rm eff}$ is induced by the diagrams with exchange of a virtual Δ between lepton and Higgs pairs.

III. An interaction of lepton-Higgs pairs with heavy Majorana triplet fermion Σ_R. The diagrams with exchange of a virtual Σ_R between the lepton-Higgs pairs induce the effective Lagrangian $\mathcal{L}_{\rm eff}$.

Models with interactions I, II and III are called type I(standard), type II and type III seesaw models

We will discuss now neutrino oscillation phenomenon
For this phenomenon uncertainty relations are important

Uncertainty relations
are based on the Cauchy inequality

$$|a^{+}b|^{2} \leq (a^{+}a)(b^{+}b)$$

where a and b are any vectors
Consider vectors $A|a\rangle$ and $B|a\rangle$ where $|a\rangle$ is any state and $A = A^{\dagger}$ and $B = B^{\dagger}$ are hermitian operators
Cauchy inequality takes the form

$$|\langle a|AB|a\rangle|^{2} \leq \langle a|A^{2}|a\rangle\langle a|B^{2}|a\rangle$$

We have

$$AB = \frac{1}{2}[A, B]_{-} + \frac{1}{2}[A, B]_{+}$$

Taking into account that $\langle a|AB|a\rangle^{*} = \langle a|BA|a\rangle$
we conclude that $\langle a|[A, B]_{-}|a\rangle$ is imaginary and $\langle a|[A, B]_{+}|a\rangle$ is real

From Cauchy inequality we have in this case

$$\langle a|A^2|a\rangle\langle a|B^2|a\rangle \geq \frac{1}{4}|\langle a|[A,B]_-|a\rangle|^2$$

Let us now make the change $A \to A - \overline{A}$ and $B \to B - \overline{B}$

$\overline{A} = \langle a|A|a\rangle$ is the average value of A

We have inequality

$$\Delta A \ \Delta B \geq \frac{1}{2}|\langle a|[A,B]_-|a\rangle|$$

$\Delta A = \sqrt{\langle a|(A-\overline{A})^2|a\rangle}$ is the standard deviation

Nontrivial constraints for noncommuting operators

If we know commutator the inequality takes the universal form (the same form for any states $|a\rangle$)

For example, for operators p and q with $[p.q] = \frac{1}{i}$ we have

$$\mathrm{Heisenberg \ \ uncertainty \ \ relation} \ \ \Delta \mathrm{p} \ \Delta \mathrm{q} \geq \frac{1}{2}$$

Less familiar

Time-energy uncertainty relation $\Delta E \; \Delta t \geq 1$

Exist different interpretation of this relation

We will consider Mandelstam-Tamm time-energy uncertainty relation

In the Heisenberg representation states do not depend on t. Operators depend on t and satisfy the equation

$$i\frac{\partial O(t)}{\partial t} = [O(t), H]$$

H is the total Hamiltonian. Commutator $[O(t), H]$ determines derivative of the operator

We have inequality

$$\Delta E \; \Delta O(t) \geq \frac{1}{2}|\frac{d}{dt}\overline{O}(t)| \quad \overline{O}(t) = \langle a|O(t)|a\rangle$$

For stationary states ($H|a\rangle = E|a\rangle$) $\Delta E = 0$ and $\frac{d}{dt}\overline{O}(t)| = 0$ no constraints

Nontrivial constraints only in the case of nonstationary states

Taking into account that ΔE does not depend on t we have

$$\Delta E \int_0^{\Delta t} \Delta O(t) dt \geq \frac{1}{2} |\overline{O}(\Delta t) - \overline{O}(0)|$$

From this inequality

$$\Delta E \ \Delta t \geq \frac{1}{2} \frac{|\overline{O}(\Delta t) - \overline{O}(0)|}{\Delta O(\bar{t})}$$

For the time interval Δt during which the state of the system is significantly changed ($\overline{O}(t)$ is changed on the value which is characterized by the standard deviation), rhs is of the order of one

We come to the Mandeshtam-Tamm time-energy uncertainty relation

$$\Delta E \ \Delta t \geq 1$$

QFT basics of neutrino oscillations

I. CC lepton current

$$j_{\alpha}^{CC}(x) = 2 \sum_{l=e,\mu,\tau} \bar{\nu}_{lL}(x)\, \gamma_{\alpha}\, l_L(x)$$

CC interaction is responsible for production and detection of neutrinos

II. Mixing relation for neutrino fields

$$\nu_{lL}(x) = \sum_{i=1}^{3} U_{li}\, \nu_{iL}(x)$$

$\nu_i(x)$ is the field of neutrino (Majorana or Dirac) with mass m_i

The main problem is how to obtain from these field-theoretical relations observable neutrino transition probabilities. There are many different opinions. A lot of discussions. Hundreds of lengthy papers

The physics of neutrino oscillations is the same as physics of $B^0 \rightleftarrows \overline{B}^0$, $K^0 \rightleftarrows \overline{K}^0$, etc oscillations

The formalism of $B^0 \rightleftarrows \overline{B}^0$ etc oscillations was confirmed by high-precision B-factory and other experiments

States with definite masses and widths (eigenstates of the total Hamiltonian)

$$|B_{H,L}\rangle = p|B^0\rangle \mp q|B^0\rangle$$

p and q are mixing parameters, $|B^0\rangle$ and $|\overline{B}^0\rangle$ are flavor states (eigenstates of the Hamiltonian of the strong (and electromagnetic) interaction)

We have the mixing relation

$$|B^0\rangle = \frac{1}{2p}(|B_H\rangle + |B_L\rangle), \quad |\overline{B}^0\rangle = \frac{1}{2q}(-|B_H\rangle + |B_L\rangle)$$

Basics of $B^0 \rightleftarrows \overline{B}^0$ etc oscillations

I. In strong processes flavor is conserved and particles with definite flavor B^0, $\overline{B}^0$, etc are produced.

In other words: Mass difference of B_H and B_L is determined by the forth order od the electroweak interaction (box diagram). It can not be resolved in production processes. As a result a coherent superposition is produced

Flavor particles are also detected

II. Evolution of mixed flavor states is given by the Schrodinger equation

$$i\frac{\partial}{\partial t}|\Psi(t)\rangle = H\ |\Psi(t)\rangle \quad |\Psi(t)\rangle = e^{-iHt}|\Psi(0)\rangle$$

If at $t=0$ B^0 is produced, at $t>0$ we have

$$|B^0(t)\rangle = \frac{1}{2p}(e^{-i\mu_H t}|B_H\rangle + e^{-i\mu_L t}|B_L\rangle) = g_+(t)|B^0\rangle - \frac{q}{p}g_-(t)|\overline{B}^0\rangle$$

$$g_\pm(t) = \frac{1}{2}(e^{-i\mu_H t} \pm e^{-i\mu_L t}), \quad \mu_{H,L} = m_{H,L} - \frac{1}{2}\Gamma_{H,L}$$

We will discuss now neutrino oscillations

Neutrinos are produced in CC weak decays and reactions. Let us consider (in lab. system) a decay

$$a \to b + l^+ + \nu_i \quad i = 1, 2, 3$$

Sum of the states of the final particles

$$|f\rangle = \sum_i |b\ \rangle |l^+\rangle |\nu_i\rangle \langle b\ l^+ \nu_i|S|a\rangle$$

$\langle b\ l^+ \nu_i|S|a\rangle$ is the matrix element of the process. We assume, as usual, that initial and final particles have definite momenta. Neutrino momenta are p_i.

If neutrino masses are equal, we have $p_i = p_k$

Thus, we have

$$p_i \simeq p + a\frac{\Delta m^2_{1i}}{2E}$$

p is the momentum of the lightest neutrino, $E \simeq p$ is the neutrino energy and a is a constant of the order of one

From neutrino oscillation data follows that $\frac{\Delta m^2_{1i}}{E^2} \leq 10^{-17}$. Thus, in the matrix element differences in neutrino momenta can be safely neglected

Lepton part of the matrix element

$$U^*_{li}\bar{u}_L(p_i)\gamma_\alpha u(-p_l) \simeq U^*_{li}\bar{u}_L(p)\gamma_\alpha u(-p_l)$$

For the matrix element we have

$$\langle b\ l^+\nu_i|S|a\rangle \simeq U^*_{li}\ \langle b\ l^+\nu_l|S|a\rangle_{SM}$$

$\langle b\ l^+\nu_l|S|a\rangle_{SM}$ is the SM matrix element of the process of production of flavor neutrino ν_l in the process $a \to b + l^+ + \nu_l$)

Dependence on i only in U^*_{li}

The final state take the form

$|f\rangle = \sum_l |b\ \rangle |l^+\rangle |\nu_l\rangle \langle b\ l^+ \nu_l|S|a\rangle_{SM}$

Here

$$|\nu_l\rangle = \sum_{i=1}^{3} U_{li}^*\ |\nu_i\rangle \quad (l = e, \mu, \tau)$$

We will consider now possibilities to reveal neutrino masses from the point of view of the Heisenberg uncertainty relation

$$(\Delta p)_{\mathrm{QM}} \simeq \frac{1}{d}$$

where d characterizes a microscopic dimension of a source

Difference of momenta of neutrinos with different masses

$|p_k - p_i| \simeq \frac{|\Delta m_{ik}^2|}{2E} = \frac{1}{L_{ik}^0}$

L_{ik}^0 is macroscopic distance: for the reactor neutrinos $L_{12}^0 \simeq 10$ km.

For the accelerator neutrinos $L_{23}^0 \simeq 100$ km

Thus, $L^0_{ik} \gg d$ and $|p_k - p_i| \ll (\Delta p)_{\mathrm{QM}}$

It follows from uncertainty relation that it is impossible to reveal different neutrino masses for neutrinos with energies relevant for neutrino oscillation experiments

The state of the flavor neutrino ν_l $(l = e, \mu, \tau)$

I. Coherent superposition of the states of neutrinos with different masses

II. Do not depend on process

III. Orthogonal and normalized

$$\langle \nu_{l'} | \nu_l \rangle = \delta_{l'l}$$

What happen with the state of the produced flavor neutrino? As in the case of $B^0 - \bar{B}^0$ we assumed first that this state is an initial state ($t = 0$) in the Schrodinger equation

At the time t for the neutrino state we have

$$|\Psi(t)\rangle_{\nu_l} = e^{-iH_0 t} \sum_i |\nu_i\rangle \ U_{li}^* = \sum_i |\nu_i\rangle e^{-iE_i t} \ U_{li}^*$$

Thus, neutrino state is a superposition of states with different energies (nonstationary state)

Neutrinos are detected via observation of weak processes in which flavor neutrinos are participating. We have

$$|\Psi(t)\rangle_{\nu_l} = \sum_{l_1} |\nu_{l_1}\rangle (\sum_i U_{l_1 i} \ e^{-iE_i t} \ U_{li}^*)$$

Let us consider the matrix element of the process $\nu_{l_1} + N \to l' + X$

Matrix element of the process

$$\langle l' \; X|S|\nu_{l_1} \; N\rangle = U^*_{l_1 i}\langle l' \; X|S|\nu_{l_1} \; N\rangle$$

Neglecting $\frac{\Delta m^2_{ik}}{2E^2}$ we have

$$\langle l' \; X|S|\nu_{l_1} \; N\rangle \simeq U_{l' i}\langle l' \; X|S|\nu_{l'} \; N\rangle_{SM}$$

Taking into account unitarity of the mixing matrix
$(\sum_i U^*_{l_1 i} U_{l' i} = \delta_{l_1 l'})$ we find

$$\langle l' \; X|S|\nu_{l_i} \; N\rangle \simeq \delta_{l_1 l'}\langle l' \; X|S|\nu_{l'} \; N\rangle_{SM}$$

1. In neutrino production and detection processes the lepton flavor is conserved.
2. Matrix elements of these processes are given by the Standard Model and do not depend on any characteristics of individual massive neutrinos.

To the chain of the processes $a \to b + l^{+} + \nu_{l} \quad \nu_{l} \to \nu_{l'}$
$\nu_{l'} + N \to l' + X$

corresponds the product

$$\langle b\ l^{+}\nu_{l}|S|a\rangle_{SM} \quad \left(\sum_{i} U_{l'i}\ e^{-iE_{i}t}\ U_{li}^{*}\right) \quad \langle l'\ X|S|\nu_{l'}\ N\rangle_{SM}$$

Only amplitude of the transition $\nu_{l} \to \nu_{l'}$ depends on the properties of massive neutrinos (mass-squared differences and mixing angles)

The factorization is based on the Heisenberg uncertainty relation

The probability of the transition $\nu_l \to \nu_{l'}$

$$P(\nu_l \to \nu_{l'}) = |\sum_i U_{l'i}\, e^{-iE_i\ t}\, U^*_{li}|^2 = |\delta_{l'l} + \sum_{i \neq k} U_{l'i}\, (e^{-i(E_i - E_k)\ t} - 1)\, U^*_{li}|^2$$

The transition take place if

$$|E_k - E_i|\ t \geq 1$$

This is Mandelstam-Tamm time-energy uncertainty relation
t is the time interval during which the transition happens,
$|E_k - E_i|$ is the energy uncertainty of the neutrino state
Space-momentum and time-energy uncertainty relations are the basis for neutrino oscillations

Flavor state $|\nu_l\rangle$ is characterized by (one) momentum $\vec{p}$

Due to relativistic invariance $p_i^2 = m_i^2$ and

$E_i = \sqrt{p^2 + m_i^2} \simeq E + \frac{m_i^2}{2E}$

For ultra relativistic neutrinos the time of the propagation of neutrino signal is equal to distance between neutrino source and detector

$$t = L$$

We come to the standard expression for the neutrino transition probability

$$P(\nu_l \to \nu_{l'}) = |\delta_{l'l} + \sum_{i \neq k} U_{l'i}\ (e^{-i\frac{\Delta m_{ik}^2}{2E}\ L} - 1)\ U_{li}^*|^2$$

In the case of the evolution in time (Schrodinger equation) transition probability depends on t and the relation $t = x$ is used

In some (many) papers on neutrino oscillations evolution in x and t is considered with evolution operator e^{-iPx} ($P^{\alpha} = (H.\vec{P})$ is the total momentum, $x^{\alpha} = (t.\vec{x})$ is space-time coordinate

Let us assume that at the space-time point $x = 0$ flavor neutrino ν_l is produced and the vector of the flavor neutrino is given

$$|\nu_l\rangle = \sum_i U^*_{li} |\nu_i\rangle$$

$|\nu_i\rangle$ is the state of neutrino with mass m_i and 4-momentum p_i

For the neutrino state at the point x we have

$$|\nu_l\rangle_x = e^{-iPx} |\nu_l\rangle = \sum_i e^{-ip_i x} U^*_{li} |\nu_i\rangle = \sum_{l'} |\nu_{l'}\rangle \left(\sum_i U_{l'i} e^{-ip_i x} U^*_{li} \right)$$

Transition probability $\nu_l \to \nu_{l'}$ is given

$$P(\nu_l \to \nu_{l'}) = \sum_{i,k} U_{l'i} U^*_{l'k} e^{-i(p_i - p_k)x} U^*_{li} U_{lk}$$

Assume $\vec{p_i} = p_i \vec{k}$ ($\vec{k}$ is an unit vector)

The phase difference at the distance L after the time t

$$(p_i - p_k)x = (p_i - p_k)L - (E_i - E_k)t, \quad i \neq k$$

For $E_i \gg m_i$ we have $p_i = E_i - \frac{m_i^2}{2E}$

The phase difference

$$(p_i - p_k)x = \frac{\Delta m^2_{ik}}{2E} L - (E_i - E_k)(t - L)$$

For the ultra relativistic neutrinos $t \simeq L$ and the last term disappears

We come to the standard expression for the phase difference

$$(p_i - p_k)x = \frac{\Delta m_{ik}^2}{2E} L$$

and the standard neutrino transition probability

Nevertheless this "QFT derivation" of the transition probability is wrong

There are two reasons

1. e^{-iPx} is not operator of the evolution of states

2. The state $|\nu_l\rangle$ can not depend on x (can not be localized)

The operator e^{-iPx} is the evolution operator of fields

This follows from invariance under translations

$$x' = x + a, \quad a \text{is an arbitrary } 4-\text{vector}$$

From invariance under translations we have that states and operators are transformed as follows

$$|\Phi'\rangle = e^{-iPa}\ |\Phi\rangle, \quad O(x+a) = e^{iPa}\ O(x)\ e^{-iPa}$$

$O(x)$ is a field operator and P is the operator of the total momentum

Calculating a derivative over a we obtain the equation

$$i\ \partial_\alpha O(x) = [O(x), P_\alpha]$$

General solution of this equation

$$O(x) = e^{iPx}\ O(0)\ e^{-iPx}$$

This means that e^{-iPx} is the operator of *evolution of the field operators* (not vectors of the states)

Let us consider neutrino evolution in the framework of Quantum Mechanics

Wave function of a neutrino, produced in a CC process as a flavor neutrino ν_l, is the superposition of plane waves

$$\psi_{\nu_l}(\vec{x},t) = \sum_i U_{li}^* \; e^{i(\vec{p}_i\vec{x}-E_i t)} u^{(-1)}(p_i)$$

where $u^{(-1)}(p_i)$ satisfies the Dirac equation

$$\gamma \cdot p_i u^{(-1)}(p_i) = m_i \; u^{(-1)}(p_i)$$

Normalized amplitude of the transition $\nu_l \to \nu_{l'}$ at the macroscopic distance L after the time t

$$\mathcal{A}(\nu_l \to \nu_{l'}) = \sum_i U_{l'i} e^{i(p_i L - E_i T)} \; U_{li}^*$$

$p_i L - E_i t$ is the change of the phase of the plane wave which describes ν_i at the distance L after the time t

For the ultrarelativistic neutrinos $L = t$ and we come to the standard expression for the transition probability

So, the standard expression could be the result of the QM evolution of the mixed neutrino wave function

Let us stress that in QM approach to neutrino oscillations the notion of flavor neutrino states is not appeared.

REMARK

Evolution of states in QFT is described by the Schrodinger equation

The Dirac equation in QM

$$i\gamma^{\alpha}\partial_{\alpha}\psi(x) = m\psi(x)$$

For the "mixed" wave function

$$i\gamma^{\alpha}\partial_{\alpha}\psi_{\nu_l}(x) = \sum_i m_i U^*_{li}\; e^{i(\vec{p}_i\vec{x} - E_i t)} u^{(-1)}(p_i) \neq m\psi_{\nu_l}(x)$$

If any wave function of a particle with spin 1/2 must satisfy the Dirac equation? It is an open problem and study of neutrino oscillations can in principle answer the question

WAVE PACKET APPROACH

Let us take into account distribution of momenta of initial neutrinos due to the uncertainty relation

For the $\nu_l \to \nu_{l'}$ transition amplitude we have in this case

$$\mathcal{A}(\nu_l \to \nu_{l'}) = \sum_i U_{l'i} \int e^{-i(\vec{p'_i}\vec{x} - E(\vec{p'}_i^2))t}\ f(\vec{p'_i} - \vec{p_i})\ d^3p'\ U^*_{li},$$

$E(\vec{p'}_i^2) = \sqrt{\vec{p'}_i^2 + m_i^2}$. The function $f(\vec{p'_i} - \vec{p_i})$ has a sharp maximum at the point $\vec{p'_i} = \vec{p_i}$. We assume also that

$$\overline{|\vec{p'_i} - \vec{p_i}|} \ll p_i.$$

We have the expansion

$$E(\vec{p'_i}^2) \simeq E_i + (\vec{p'_i} - \vec{p_i}) \cdot \vec{v_i}$$

$$v_i^k = \frac{\partial E(p'^2_i)}{\partial p'^k_i}|_{p'^k_i = p^k_i} = \frac{p_i^k}{E_i}$$

For the amplitude of the transition $\nu_l \to \nu_{l'}$ we find

$$\mathcal{A}(\nu_l \to \nu_{l'}) = \sum_i U_{l'i} e^{-i\vec{p}_i\vec{x} - iE_i t} g(\vec{x} - \vec{v}_i t)\ U^*_{li}$$

where

$$g(\vec{x} - \vec{v}_i t) = \int e^{i\vec{q}\ (\vec{x} - \vec{v}_i t)}\ f(\vec{q})\ d^3q.$$

From the amplitude in the plane wave approximation $\mathcal{A}(\nu_l \to \nu_{l'})$ differs by the factor which (due to relation between momentum and energy) depends on $\vec{x} - \vec{v}_i t$

Usually it is assumed that the function $f(\vec{q})$ has Gaussian form

$$f(\vec{q}) = N e^{-\frac{q^2}{4\sigma_p^2}}$$

σ_p is a width

After the integration we find

$$g(\vec{x}-\vec{v}_i t) = N(\frac{\pi}{\sigma_x^2})^{3/2}\ e^{-\frac{(\vec{x}-\vec{v}_i t)^2}{4\sigma_x^2}}$$

$$\sigma_x^2 = \frac{1}{4\sigma_p^2}$$

σ_x characterizes spacial width of the wave packet

x and t are macroscopic quantities much larger than σ_x

Thus, for ultrarelativistic neutrinos the amplitude $g(\vec{x}-\vec{v}_i t)$ provides the equality $x \simeq t$

In the wave packet approach the probability of the transition $\nu_l \to \nu_{l'}$ is determined as integrated over the time quantity (assuming that the time is not measured)

$$P(\nu_l \to \nu_{l'}) = \int_{-\infty}^{+\infty} |\mathcal{A}(\nu_l \to \nu_{l'})|^2 dt$$

This integration provide the equality $x \simeq t$ and allows to calculate small terms due to σ_x

Integrated transition probability

$$P(\nu_l \to \nu_{l'}) = N^2(\frac{\pi}{\sigma_x^2})^3 \sum_{i,k} U_{l'i} U^*_{l'k} e^{i(p_i - p_k)x} U^*_{li} U_{lk} \int_{-\infty}^{+\infty} e^{-\frac{t^2}{2\sigma_x^2}} dt \, e^{-A}.$$

Here

$$A = -i(E_i - E_k)x + \frac{1}{2\sigma_x^2} \left(\frac{\Delta m_{ik}^2}{4E^2}\right)^2 x^2 + \frac{1}{2}\sigma_x^2 \xi^2 \frac{\Delta m_{ik}^2}{2E}$$

We took into account that

$$E_i = E + \xi \frac{\Delta m_{1i}^2}{2E},$$

The coefficient ξ is of the order of one

From the first term for the expression for A follows that the wave packet approach provides the relation $t \simeq x$

We find the standard expression for the oscillation phase

$$[(p_i - p_k) - (E_i - E_k)]x = \frac{\Delta m_{ik}^2}{2E} x.$$

Additional terms

The second term of A is due to the relation

$$(v_i - v_k) = \frac{\Delta m^2_{ik}}{2E^2}$$

Thus, this term is different from zero because neutrino with different masses have different velocities. For the ultra relativistic neutrinos this term is extremely small.

As we discussed before, due to the condition

$$|p_i - p_k| \simeq |\frac{\Delta m^2_{ik}}{2E}| \ll \frac{1}{\sigma_x}$$

flavor neutrino states are coherent.

From this condition follows that the third term of the expression A is also small.

The normalized $\nu_l \to \nu_{l'}$ transition probability

$$\mathcal{P}(\nu_l \to \nu_{l'}) = \left(\sum_{i,k} U_{l'i} U^*_{l'k} e^{i\frac{\Delta m^2_{ik}}{2E}L} U^*_{li} U_{lk} \right) e^{-(\frac{L}{L^{ik}_{\mathrm{coh}}})^2} \; e^{-2\pi^2\xi^2(\frac{\sigma_x}{L^{ik}_{\mathrm{o}}})^2}.$$

Here $L = x$ is the distance between source and detector and in the brackets the standard expression for the transition probability

$$L^{ik}_{\mathrm{coh}} = \frac{4\sqrt{2}\sigma_x E^2}{|\Delta m^2_{ik}|}, \quad L^{ik}_{\mathrm{o}} = 4\pi \frac{E}{|\Delta m^2_{ik}|}$$

are coherence and oscillation lengths

The coherence length characterizes the time interval (distance) at which the distance between ν_i and ν_k becomes comparable with the size of the wave packet:

$$|(v_i - v_k)| \; L^{ik}_{\mathrm{coh}} \sim \sigma_x.$$

We have

$$L^{ik}_{\rm coh} = \frac{E}{\sqrt{2}\pi\sigma_p} \gg L^{ik}_{\rm o}$$

Thus, the coherence length is much larger than the oscillation length

The second multiplier in the expression for the transition probability is practically equal to one for any conceivable neutrino oscillation experiments

The effect of the decoherence could be important only for huge cosmological distances.

The condition of the coherence of the states of neutrinos with different masses

$$L^{ik}_{\rm o} \gg \sigma_x.$$

The third multiplier in the expression for the transition probability is also practically equal to one

Conclusion

The wave packet approach bring us to the standard plane wave expression for the transition probability

The wave packet approach allows to justify the correctness of the plane wave approximation and the relation $x = t$