Theory of Neutrino Masses and Mixing

Walter Grimus
(University of Vienna)

IV International Pontecorvo Neutrino Physics School

September 26 – October 6, 2010, Alushta, Crimea

Motivation

Motivation for horizontal symmetries:

Gatto, Sartori, Tonin; Cabibbo, Maiani (1968):

$$\sin\theta_c \simeq \sqrt{\frac{m_d}{m_s}}$$

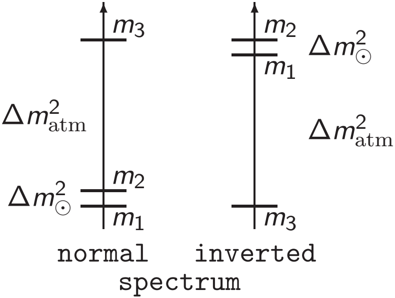
Harrison, Perkins, Scott (2002):

$$U \simeq \begin{pmatrix} 2/\sqrt{6} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix} e^{i\hat{\beta}} \equiv U_{\mathrm{HPS}}\, e^{i\hat{\beta}}$$

Mass spectra of charged fermions (up quarks, down quarks, charged leptons) strongly hierarchical!
Mixing angles = functions of quark mass ratios?
Mass spectrum of neutrinos: either completely different or hierarchy not so pronounced
Mixing angles "pure numbers"?

Motivation

Neutrino mass spectrum non-degenerate:



Hierarchical spectrum: normal with $m_1 \to 0$

$\Delta m^2_{\mathrm{atm}}/\Delta m^2_{\odot} \sim 30 \Rightarrow m_3/m_2 \simeq \sqrt{\Delta m^2_{\mathrm{atm}}/\Delta m^2_{\odot}} \sim 5 \div 6$

Inverted hierarchy: inverted with $m_3 \to 0$

Smallest ν mass m_s:

$m_s = m_1$ for normal, $m_s = m_3$ for inverted

Neutrino mass terms and parameter counting

Assumptions:

- Majorana neutrinos
- Charged-lepton mass matrix diagonal
  (for the purpose of parameter counting)

Majorana neutrino mass term:

$$\mathcal{L}_{\mathrm{Maj}} = \frac{1}{2}\nu_L^T C^{-1} \mathcal{M}_\nu \nu_L + \mathrm{H.c.}$$

$\mathcal{M}_\nu$ complex, symmetric!

Neutrino mass terms and parameter counting

Assumptions:

- Majorana neutrinos
- Charged-lepton mass matrix diagonal (for the purpose of parameter counting)

Majorana neutrino mass term:

$$\mathcal{L}_{\mathrm{Maj}} = \frac{1}{2}\nu_L^T C^{-1} \mathcal{M}_\nu \nu_L + \mathrm{H.c.}$$

$\mathcal{M}_\nu$ complex, symmetric!

Theorem (Schur)

$\mathcal{M}_\nu^T = \mathcal{M}_\nu \Rightarrow \exists$ *unitary matrix* U *with*
$U^T \mathcal{M}_\nu U = diag(m_1, m_2, m_3)$ *with* $m_j \geq 0$

PMNS or lepton mixing matrix U (modulo phase multiplications from the left)

Parameterization of the mixing matrix:

$$\boxed{U = e^{i\hat{\alpha}} U_{23} U_{13} U_{12} \,\mathrm{diag}\left(1, e^{i\beta_1}, e^{i\beta_2}\right)}$$

with $e^{i\hat{\alpha}} = \mathrm{diag}\left(e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_3}\right)$
$e^{i\hat{\alpha}}$ are unphysical phases in charged current interaction
(can be absorbed into the charged lepton fields)

$$\begin{aligned} U_{23} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \\ U_{13} &= \begin{pmatrix} c_{13} & 0 & s_{13} e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13} e^{i\delta} & 0 & c_{13} \end{pmatrix} \\ U_{12} &= \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Neutrino mass terms and parameter counting

$$U_{23} U_{13} U_{12} =$$

$$\begin{pmatrix} c_{12} c_{13} & s_{12} c_{13} & s_{13} e^{-i\delta} \\ -s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i\delta} & c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i\delta} & s_{23} c_{13} \\ s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i\delta} & -c_{12} s_{23} - s_{12} c_{23} s_{13} e^{i\delta} & c_{23} c_{13} \end{pmatrix}$$

Conventions:

- $0^\circ \leq \theta_{ij} \leq 90^\circ$
- $0^\circ \leq \delta < 360^\circ$
- $m_1 < m_2$ with $\Delta m^2_\odot = m_2^2 - m_1^2$

Physical phases:

CKM-type phase δ, Majorana phases β_1, β_2

Neutrino mass terms and parameter counting

Parameter counting:

9 physical parameters: 3 masses, 3 angles, 3 phases

Physical parameters in $\mathcal{M}_\nu$:

$6 \times 2 = 12$ real parameters in $\mathcal{M}_\nu$

first line and column can be made real by phase transformation $\Rightarrow$

9 real physical parameters in $\mathcal{M}_\nu$

Discrete physical parameter:

$\operatorname{sign}(m_3^2 - m_1^2) \Rightarrow$ normal vs. inverted mass spectrum

Diagonalization of the neutrino mass matrix:

$$U^T \mathcal{M}_\nu U = \mathrm{diag}\,(m_1, m_2, m_3) \equiv \hat{m} \Rightarrow \mathcal{M}_\nu U = U^* \hat{m}$$

$$U = (u_1, u_2, u_3) \Rightarrow \boxed{\mathcal{M}_\nu u_j = m_j u_j^*}$$

Diagonalization of the neutrino mass matrix:

$$U^T \mathcal{M}_\nu U = \mathrm{diag}\,(m_1, m_2, m_3) \equiv \hat{m} \Rightarrow \mathcal{M}_\nu U = U^* \hat{m}$$

$$U = (u_1, u_2, u_3) \Rightarrow \boxed{\mathcal{M}_\nu u_j = m_j u_j^*}$$

Note:

- In general, u_j is not an eigenvector of $\mathcal{M}_\nu$, only for real u_j.
- If λ is an eigenvalue of $\mathcal{M}_\nu$, then $|\lambda|$ is in general NOT a neutrino mass.
- However, $\mathcal{M}_\nu^\dagger \mathcal{M}_\nu u_j = m_j^2 u_j$.

Basics

Definition of a group G:

G is a set with a multiplication rule

0. **Closure:**
   $g_1 \in G,\ g_2 \in G \Rightarrow g_1 g_2 \in G$
1. **Associativity:**
   $(g_1 g_2) g_3 = g_1 (g_2 g_3)$
2. **Unit element:**
   It exists $e \in G$ such that $eg = g\ \forall\, g \in G$.
3. **Inverse element:**
   $\forall\, g \in G$ it exists $g^{-1} \in G$ such that $g^{-1} g = e$.

Definition of a group G:

G is a set with a multiplication rule

0. **Closure:**
   $g_1 \in G,\ g_2 \in G \Rightarrow g_1 g_2 \in G$
1. **Associativity:**
   $(g_1 g_2) g_3 = g_1 (g_2 g_3)$
2. **Unit element:**
   It exists $e \in G$ such that $eg = g \ \forall\, g \in G$.
3. **Inverse element:**
   $\forall\, g \in G$ it exists $g^{-1} \in G$ such that $g^{-1} g = e$.

Remarks:

- left inverse = right inverse,
  left unit element = right unit element,
  inverse and unit element are unique
- associativity always fulfilled for mappings
  (permutations, matrices, ...)

Group representations:

- Vector space $\mathcal{V}$ over $\mathbb{C}$
- $L(\mathcal{V})$ = set of linear operators on $\mathcal{V}$
- $D: \; G \to L(\mathcal{V})$ such that $D(g_1 g_2) = D(g_1)D(g_2)$
- $D(e) = \mathbb{1}$

Group representations:

- Vector space $\mathcal{V}$ over $\mathbb{C}$
- $L(\mathcal{V})$ = set of linear operators on $\mathcal{V}$
- $D: \; G \to L(\mathcal{V})$ such that $D(g_1 g_2) = D(g_1) D(g_2)$
- $D(e) = \mathbb{1}$

Unitary representation:

$\mathcal{V}$ with scalar product $\langle x | y \rangle$ and $\langle D(g) x | D(g) y \rangle = \langle x | y \rangle \; \forall \, g$

Group representations:

- Vector space $\mathcal{V}$ over $\mathbb{C}$
- $L(\mathcal{V})$ = set of linear operators on $\mathcal{V}$
- $D: \ G \to L(\mathcal{V})$ such that $D(g_1 g_2) = D(g_1)D(g_2)$
- $D(e) = \mathbb{1}$

Unitary representation:

$\mathcal{V}$ with scalar product $\langle x|y\rangle$ and $\langle D(g)x|D(g)y\rangle = \langle x|y\rangle \;\forall\, g$

Irreducible representation: "irrep"

$\mathcal{V}$ does *not* have any non-trivial subspace $\mathcal{W}$ such that $D(g)\mathcal{W} = \mathcal{W} \;\forall\, g$

Basics

Unitary representations can be decomposed into irreps!

Irreps are the smallest building blocks of representations

$$D(g) \to \begin{pmatrix} D_1(g) & 0 & 0 & \cdots \\ 0 & D_2(g) & 0 & \cdots \\ 0 & 0 & D_3(g) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Basics

Definition: $H \subseteq G$ is a normal subgroup if $gHg^{-1} = H \; \forall g \in G$

Basics

Definition: $H \subseteq G$ is a normal subgroup if $gHg^{-1} = H \; \forall g \in G$

Definition: H normal subgroup of G, then the factor group G/H consists of the cosets $\{H, Hg_1, Hg_2 \ldots\}$ with the multiplication rule (H g) (Hg') = Hgg'

Basics

Definition: $H \subseteq G$ is a normal subgroup if $gHg^{-1} = H \; \forall g \in G$

Definition: H normal subgroup of G, then the factor group G/H consists of the cosets $\{H, Hg_1, Hg_2 \ldots\}$ with the multiplication rule (H g) (Hg') = Hgg'

Definition: g_1 is conjugate to g_2 if $\exists g \in G$ such that $gg_1g^{-1} = g_2$

Basics

Definition: $H \subseteq G$ is a normal subgroup if $gHg^{-1} = H \; \forall g \in G$

Definition: H normal subgroup of G, then the factor group G/H consists of the cosets $\{H, Hg_1, Hg_2 \ldots\}$ with the multiplication rule (H g) (Hg') = Hgg'

Definition: g_1 is conjugate to g_2 if $\exists g \in G$ such that $gg_1g^{-1} = g_2$

Definition: "g_1 conjugate to g_2" defines an equivalence relation $\Rightarrow$ the sets of equivalent elements are called conjugacy classes.

Basics

Definition: $H \subseteq G$ is a normal subgroup if $gHg^{-1} = H \;\forall g \in G$

Definition: H normal subgroup of G, then the factor group G/H consists of the cosets $\{H, Hg_1, Hg_2 \ldots\}$ with the multiplication rule (H g) (Hg') = Hgg'

Definition: g_1 is conjugate to g_2 if $\exists g \in G$ such that $gg_1g^{-1} = g_2$

Definition: "g_1 conjugate to g_2" defines an equivalence relation $\Rightarrow$ the sets of equivalent elements are called conjugacy classes.

Remarks:
$\{e\}$ is a class.
A normal subgroup consists of complete conjugacy classes of G.
Let H be a proper normal subgroup of G $\Rightarrow$

- The mapping $f : g \in G \to Hg \in G/H$ is a homomorphism, i.e., $f(g)f(g') = f(gg')$.
- Any representation D of G/H induces naturally a representation $\bar{D}$ of G via $\bar{D}(g) \equiv D(Hg)$.

Basics

Direct product: $G \times G'$ with multiplication law
$(g_1, g_1')(g_1, g_1') = (g_1 g_1', g_2 g_2')$
E.g. $S_3 \times \mathbb{Z}_2$

Semidirect product: $H \rtimes_\phi G$
G acts on H via the homomorphism $\phi: \; G \to \mathrm{Aut}(H)$
Multiplication law: $(h_1, g_1)(h_2, g_2) = (h_1 \, \phi(g_1) h_2, g_1 g_2)$

Basics

Direct product: $G \times G'$ with multiplication law
$(g_1, g_1')(g_1, g_1') = (g_1 g_1', g_2 g_2')$
E.g. $S_3 \times \mathbb{Z}_2$

Semidirect product: $H \rtimes_\phi G$
G acts on H via the homomorphism $\phi: \; G \to \mathrm{Aut}(H)$
Multiplication law: $(h_1, g_1)(h_2, g_2) = (h_1\, \phi(g_1) h_2, g_1 g_2)$

Remarks: If $\phi = \mathrm{id} \Rightarrow H \rtimes_\phi G \equiv H \times G$
Useful question for model builiding: Can a group be decomposed into a semidirect product?

Semidirect products are ubiquitous!

Theorem

Group S, H proper normal subgroup of S, G subgroup of S with following properties:

1. $H \cap G = \{e\}$,
2. *every element* $s \in S$ *can be written as* $s = hg$ *with* $h \in H$, $g \in G$.

Then the following holds:

- $S \cong H \rtimes_\phi G$ *with* $\phi(g)h = ghg^{-1}$,
- *decomposition* $s = hg$ *is unique,*
- $S/H \cong G$.

$$s_1 s_2 = (h_1 g_1)(h_2 g_2) = (h_1 g_1 h_2 g_1^{-1})(g_1 g_2)$$

Symmetries in the Lagrangian vs. symmetry groups:

Multiplet of (fermion) fields $\psi_1, \ldots \psi_r$

$$\mathcal{L} = i \sum_{j=1}^{r} \bar{\psi}_j \gamma^\mu \partial_\mu \psi_j + \cdots$$

Symmetries $\psi_j \to A^{(p)}_{jk} \psi_k$ $(p = 1, \ldots, N_{\mathrm{gen}})$ of $\mathcal{L}$

$A^{(p)}$ $(p = 1, \ldots, N_{\mathrm{gen}})$ unitary matrices!

Basics

Symmetries in the Lagrangian vs. symmetry groups:

Multiplet of (fermion) fields $\psi_1, \ldots \psi_r$

$$\mathcal{L} = i \sum_{j=1}^{r} \bar{\psi}_j \gamma^\mu \partial_\mu \psi_j + \cdots$$

Symmetries $\psi_j \to A^{(p)}_{jk} \psi_k$ $(p = 1, \ldots, N_{\mathrm{gen}})$ of $\mathcal{L}$

$A^{(p)}$ $(p = 1, \ldots, N_{\mathrm{gen}})$ unitary matrices!

Two approaches to symmetries and Lagrangians:

- $\mathcal{L} \Rightarrow$ imposing symmetries $A^{(p)}$ on $\mathcal{L} \Rightarrow$ the $A^{(p)}$ represent generators of a group $G \Rightarrow$ representation of $G \Rightarrow G$
- Group $G \Rightarrow$ representations $\Rightarrow$ multiplets of fields $\Rightarrow \mathcal{L}$

Infinite vs. finite groups

Infinite groups: number of elements is infinite

- Infinitely many inequivalent irreps
- Non-compact simple Lie groups G possess no finite-dimensional unitary irreps apart from the trivial reps $g \to \mathbb{1} \; \forall g \in G$

Infinite vs. finite groups

Infinite groups: number of elements is infinite

- Infinitely many inequivalent irreps
- Non-compact simple Lie groups G possess no finite-dimensional unitary irreps apart from the trivial reps $g \to \mathbb{1}\ \forall g \in G$

Finite groups:

- Finite number of inequivalent irreps
- All irreps can be considered unitary
- Since $\mathrm{ord}\, G$ is finite, all numbers concerning properties of the group and its irreps are finite as well $\Rightarrow$ extremely useful relations (totally lacking in infinite groups)

Infinite vs. finite groups

Infinite groups: number of elements is infinite

- Infinitely many inequivalent irreps
- Non-compact simple Lie groups G possess no finite-dimensional unitary irreps apart from the trivial reps $g \to \mathbb{1}\ \forall g \in G$

Finite groups:

- Finite number of inequivalent irreps
- All irreps can be considered unitary
- Since $\operatorname{ord} G$ is finite, all numbers concerning properties of the group and its irreps are finite as well $\Rightarrow$ extremely useful relations (totally lacking in infinite groups)

Remarks:

$\operatorname{ord} G \equiv \#$ elements of G

Irreps of $U(1)$: $e^{i\alpha} \to e^{in\alpha}$ with $n \in \mathbb{Z}$

Theorems on finite groups

Subgroups:

Theorem (Lagrange)

H subgroups of $G \Rightarrow \mathrm{ord}\, H$ is a divisor of $\mathrm{ord} G$

Subgroups:

Theorem (Lagrange)

H subgroups of $G \Rightarrow \mathrm{ord}\, H$ is a divisor of $\mathrm{ord} G$

Proof: $g_1, g_2 \in G$, consider the sets Hg_1 and Hg_2.
Suppose $Hg_1 \cap Hg_2$ is not empty $\Rightarrow \exists g \in Hg_1 \cap Hg_2$
$\Rightarrow g = hg_1 = h' g_2$ with $h, h' \in H$ and $g_2 = h'^{-1} hg_1$
$\Rightarrow Hg_1 = Hg_2$.
Consequently, either $Hg_1 = Hg_2$ or $Hg_1 \cap Hg_2 = \varnothing \Rightarrow$
G can be written as
$G = H \cup Hg_1 \cup \cdots Hg_{n-1}$ with empty intersections
$\Rightarrow \mathrm{ord}\, G / \mathrm{ord}\, H = n$ Q.E.D.

Theorems on finite groups

Subgroups:

Theorem (Lagrange)

H subgroups of $G \Rightarrow \text{ord}\, H$ is a divisor of $\text{ord}\, G$

Proof: $g_1,\, g_2 \in G$, consider the sets Hg_1 and Hg_2.
Suppose $Hg_1 \cap Hg_2$ is not empty $\Rightarrow \exists g \in Hg_1 \cap Hg_2$
$\Rightarrow g = hg_1 = h' g_2$ with $h,\, h' \in H$ and $g_2 = h'^{-1} hg_1$
$\Rightarrow Hg_1 = Hg_2$.
Consequently, either $Hg_1 = Hg_2$ or $Hg_1 \cap Hg_2 = \varnothing \Rightarrow$
G can be written as
$G = H \cup Hg_1 \cup \cdots Hg_{n-1}$ with empty intersections
$\Rightarrow \text{ord}\, G / \text{ord}\, H = n$ Q.E.D.

Definition: The order of an element g is the smallest number r such that $g^r = e$

Every element $g \in G$ generates a cyclic subgroup $\mathbb{Z}_r \subseteq G$

The order of every element is a divisor of $\text{ord}\, G$

Theorems on finite groups

Theorem

$D^{(\alpha)}$ irrep of G, $\dim D^{(\alpha)} = d_\alpha$, and the index α numbers all inequivalent irreps $\Rightarrow$

$$\sum_\alpha d_\alpha^2 = ord G$$

Theorem

The number of inequivalent irreps $D^{(\alpha)}$ = number of classes of G

Finite groups used as family symmetries

Following types of finite groups usually occur:

1. Groups of permutations
2. Groups consisting of unitary matrices
3. Direct products of such groups
4. Semidirect products of such groups

Finite groups used as family symmetries

Following types of finite groups usually occur:

1. Groups of permutations
2. Groups consisting of unitary matrices
3. Direct products of such groups
4. Semidirect products of such groups

Discussion of

$$S_3, \quad S_4, \quad A_4 \equiv T, \quad D_4, \quad T'$$

S_n: group of all permutations of n objects

$$p = \begin{pmatrix} 1 & 2 & \cdots & n \\ p_1 & p_2 & \cdots & p_n \end{pmatrix}, \quad \text{ord}\, S_n = n!$$

Cycle of length r: $(n_1 \to n_2 \to n_3 \to \cdots n_r \to n_1) \equiv (n_1 n_2 n_3 \cdots n_r)$
All numbers $n_1, \ldots, n_r$ are different

Theorem

Every permutation is a unique product of cycles which have no common elements

Example: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 3 & 5 & 1 & 2 \end{pmatrix} = (145)(3)(26)$

Remarks:
Cycles which have no common element commute
A cycle which consists of only one element is identical with the unit element of S_n

Finite groups used as family symmetries

Theorem

The classes of S_n consist of the permutations with the same cycle structure

Examples:

S_3: e, $(n_1 n_2)$, $(n_1 n_2 n_3)$ $\Rightarrow$ 3 classes $\Rightarrow$ 3 inequivalent irreps

S_4: e, $(n_1 n_2)$, $(n_1 n_2 n_3)$, $(n_1 n_2 n_3 n_4)$, $(n_1 n_2)(n_3 n_4)$ $\Rightarrow$ 5 classes $\Rightarrow$ 5 inequivalent irreps

Even and odd permutations:

Every permutation of S_n is associated with an $n \times n$ permutation matrix

For instance $(123) \in S_3 \rightarrow \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = (e_2 e_3 e_1)$

In general: $p \in S_n \rightarrow M(p) = (e_{p_1} e_{p_2} \cdots e_{p_n})$

Definition: $\mathrm{sgn}(p) = \det M(p)$

Definition: even (odd) permutation with $\mathrm{sgn}(p) = +1$ (-1)

Sign of cycles:
$(n_1 n_2 n_3 \cdots n_{r-1} n_r) = (n_1 n_r)(n_1 n_{r-1}) \cdots (n_1 n_3)(n_1 n_2) \Rightarrow$
if r is even (odd) then the cycle is odd (even)

Remarks:
$p \to M(p)$ is an n-dimensional reducible representation
properties of determinant $\Rightarrow p \to \mathrm{sgn}(p)$ is a 1-dimensional irrep

Theorem

S_n has exactly two 1-dimensional irreps:
$p \to 1$ and $p \to sgn(p)$

Dimensions if irreps of S_3: $1^2 + 1^2 + d_3^2 = 6 \Rightarrow d_3 = 2$
Dimensions if irreps of S_4: $1^2 + 1^2 + d_3^2 + d_4^2 + d_5^2 = 24 \Rightarrow$
$d_3 = 2,\ d_4 = d_5 = 3$

Structure and irreps of S_3:

Generators: $h = (123)$, $g = (12)$ with $h^3 = e$, $g^2 = e$, $ghg = h^2$
Every element of S_3 can be decomposed as
$h^k g^\ell$ with $k = 0, 1, 2$, $\ell = 0, 1$

$$h \rightsquigarrow \mathbb{Z}_3,\ g \rightsquigarrow \mathbb{Z}_2 \Rightarrow$$

Structure of S_3

$S_3 \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_2$

1-dimensional irreps: $p \longrightarrow 1$, $p \longrightarrow \mathrm{sign}(p)$
correspond to irreps of $S_3/\mathbb{Z}_3 \cong \mathbb{Z}_2$

Finite groups used as family symmetries

2-dimensional irrep: $(D(h))^3 = (D(g))^2 = \mathbb{1}$
Without loss of generality: $D(h)$ diagonal $\Rightarrow$

$$h \rightarrow D(h) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \quad g \rightarrow D(g) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with $\omega = e^{2\pi i/3}$

2-dimensional irrep: $(D(h))^3 = (D(g))^2 = \mathbb{1}$
Without loss of generality: $D(h)$ diagonal $\Rightarrow$

$$h \to D(h) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \quad g \to D(g) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with $\omega = e^{2\pi i/3}$

Real version:

$$V^T \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix} V = \begin{pmatrix} \cos 120^\circ & -\sin 120^\circ \\ \sin 120^\circ & \cos 120^\circ \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

$$V^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} V = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with $V = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

Structure and irreps of S_4:

Definition: Klein's four-group

$$K = \{e,\ (12)(34),\ (13)(24),\ (14)(23)\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$$

K normal subgroup of S_4

Structure and irreps of S_4:

Definition: Klein's four-group

$$K = \{e,\ (12)(34),\ (13)(24),\ (14)(23)\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$$

K normal subgroup of S_4

Theorem

Every element of $s \in S_4$ can be uniquely decomposed as $s = kp$ with $k \in K$ and p being a permutation of the numbers 2,3,4.

Proof: Suppose $s \in S_4$ has the form $s = k_1 p_1 = k_2 p_2$
$\Rightarrow p_2 = k_1 k_2 p_1 \Rightarrow k_1 k_2 \in K$ must map 1 into 1
$\Rightarrow k_1 k_2 = e$ and $k_1 = k_2,\ p_1 = p_2$
Since ord $K \times$ ord $S_3 = 4 \times 6 = 24 =$ ord S_4
$\Rightarrow$ all elements of S_4 can be written as kp Q.E.D.

$$(k_1 p_1)(k_2 p_2) = (k_1 p_1 k_2 p_1^{-1})(p_1 p_2) \Rightarrow$$

Structure of S_4

$S_4 \cong K \rtimes S_3$

1-dimensional irreps: $p \longrightarrow 1,\ p \longrightarrow \mathrm{sign}(p)$

2-dimensional irrep: $kp \longrightarrow D_2(p)$ where D_2 is the 2-dim irrep of S_3

$$(234) \longrightarrow \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \quad (34) \longrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Example: $(12) = (12)(34)\,(34) \longrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

3-dimensional irreps:

3-dimensional representation of $K \cong \mathbb{Z}_2 \times \mathbb{Z}_2$:

$k_1 \neq k_2 \neq k_3 \neq k_1 \Rightarrow k_1 k_2 = k_2 k_1 = k_3$

$(34)^2 = e$, (34) commutes with (12)(34), etc.

$$(34) \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{aligned} (12)\,(34) &\rightarrow \mathrm{diag}\,(\ \ 1, -1, -1) \\ (13)\,(24) &\rightarrow \mathrm{diag}\,(-1, \ \ 1, -1) \\ (14)\,(23) &\rightarrow \mathrm{diag}\,(-1, -1, \ \ 1) \end{aligned} \qquad (24) \rightarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$(23) \rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

✤ Two inequivalent 3-dim irreps: $(34) \to \pm(e_1 e_3 e_2)$ ✤

Summary of S_4 irreps:

$$
s = kp \in S_4 \quad
\begin{array}{ll}
\mathbf{1}: & kp \to 1 \\
\mathbf{1}': & kp \to \mathrm{sgn}(p) \\
\mathbf{2}: & kp \to D_2(p) \\
\mathbf{3}: & kp \to A(k) M_3(p) \\
\mathbf{3}': & kp \to \mathrm{sgn}(p)\, A(k) M_3(p)
\end{array}
$$

$A\,[(12)\,(34)] = \mathrm{diag}\,(1,-1,-1)$, etc., ...
$M_3(p)$ 3×3 permutation matrix
Note: $\mathrm{sgn}(p) \equiv \det M_3(p)$

Alternating group A_n: Group of all even permutation of n objects, ord $A_n = n!/2$

Note that $S_n \cong A_n \rtimes \mathbb{Z}_2$ with $\mathbb{Z}_2$ genererated e.g. by (12)

Alternating group A_n: Group of all even permutation of n objects, ord $A_n = n!/2$

Note that $S_n \cong A_n \rtimes \mathbb{Z}_2$ with $\mathbb{Z}_2$ genererated e.g. by (12)

Theorem

A_n simple for $n \geq 5$
A_5 with 60 elements smallest simple group!

Alternating group A_n: Group of all even permutation of n objects, ord $A_n = n!/2$

Note that $S_n \cong A_n \rtimes \mathbb{Z}_2$ with $\mathbb{Z}_2$ genererated e.g. by (12)

Theorem

A_n simple for $n \geq 5$
A_5 with 60 elements smallest simple group!

Theorem (Properties of A_4)

A_4 has Klein's four-group K as proper normal subgroup
$A_4/K \simeq \mathbb{Z}_3$ with $\mathbb{Z}_3$ generated by (234)
Smallest group with a 3-dim irrep

Irreps of A_4:

$$
\begin{array}{lllll}
\mathbf{1}: & kp \longrightarrow 1 & & & \\
\mathbf{1}': & k \longrightarrow 1, & (234) \longrightarrow \omega, & (243) \longrightarrow \omega^2 \\
\mathbf{1}'': & k \longrightarrow 1, & (234) \longrightarrow \omega^2, & (243) \longrightarrow \omega
\end{array}
$$

$$
\mathbf{3}: \quad k \longrightarrow A(k),\ (234) \longrightarrow \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},\ (243) \longrightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}
$$

Remarks:
$1^2 + 1^2 + 1^2 + 3^2 = 12 = \text{ord}\, A_4$
Four classes: $\{e\}$, $\{(12)(34), (13)(24), (14)(23)\}$,
$\{(132), (124), (234), (143)\}$, $\{(123), (142), (243), (134)\}$

Dihedral groups D_n:

Definition: D_n is the group of order $2n$ generated by

$$R_n = \begin{pmatrix} \cos\frac{2\pi}{n} & -\sin\frac{2\pi}{n} \\ \sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

D_n consists of the elements $\{\mathbb{1}, R_n, \ldots, R_n^{n-1}, S, R_n S, \ldots, R_n^{n-1} S\}$
Properties: $R_n^n = S^2 = \mathbb{1}$, $SR_nS = R_n^{-1} = R_n^{n-1}$

Dihedral groups D_n:

Definition: D_n is the group of order $2n$ generated by

$$R_n = \begin{pmatrix} \cos\frac{2\pi}{n} & -\sin\frac{2\pi}{n} \\ \sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

D_n consists of the elements $\{\mathbb{1}, R_n, \ldots, R_n^{n-1}, S, R_nS, \ldots, R_n^{n-1}S\}$
Properties: $R_n^n = S^2 = \mathbb{1}$, $SR_nS = R_n^{-1} = R_n^{n-1}$

Discussion of D_4: 8 elements, R_4 rotation by 90°, S reflection at x-axis
Classes: $\{\mathbb{1}\}$, $\{-\mathbb{1}\}$, $\{\pm R_4\}$, $\{\pm S\}$, $\{\pm R_4 S\}$
Dimensions of irreps of D_4: $1^2 + d_2^2 + \cdots + d_5^2 = 8$
$\Rightarrow d_2 = d_3 = d_4 = 1,\ d_5 = 2$

Irreps of D_4:

D_4 defined via 2-dim irrep!

1-dim irreps: $S^2 = R_4^4 = \mathbb{1}$, $SR_4S = R_4^3 \Rightarrow S \to \pm 1$, $R_4 \to \pm 1$

$$\mathbf{1}^{(p,q)}: \quad S \to (-1)^p, \quad R_4 \to (-1)^q$$

Subgroups of $SO(3)$ vs. subgroups of $SU(2)$

Subgroup of $SO(3)$ $\rightsquigarrow$ subgroup of $SU(2)$

Connection between $SO(3)$ and $SU(2)$:
α = rotation angle, $\vec{n}$ = rotation axis

Every rotation induces exactly two $SU(2)$ transformations via

$$U\,\vec{\sigma}\cdot\vec{x}\,U^\dagger = \vec{\sigma}\cdot(R(\alpha,\vec{n})\vec{x}) \;\Rightarrow\; U(\alpha,\vec{n}) = \pm\left(\cos\frac{\alpha}{2}\mathbb{1} - i\sin\frac{\alpha}{2}\vec{n}\cdot\vec{\sigma}\right)$$

With this construction, for every $G \subset SO(3)$ one obtains its double-valued group (covering group) $G' \subset SU(2)$ such that $G'/G \cong \mathbb{Z}_2$, $\mathrm{ord}\, G' = 2 \times \mathrm{ord}\, G$

Finite groups used as family symmetries

Double-valued group of A_4: T'

$A_4 \subset SO(3)$ via its 3-dim faithful irrep!
In this sense, A_4 is generated by
$A = \mathrm{diag}\,(1,-1,-1) = R(180^\circ, \vec{e}_x)$,

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = R\left(-120^\circ, (\vec{e}_x + \vec{e}_y + \vec{e}_z)/\sqrt{3}\right)$$

$\Rightarrow T'$ generated by

$$U_A = i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad U_R = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi & \phi \\ -\phi^* & \phi^* \end{pmatrix} \quad \text{with} \quad \phi = e^{i\pi/4}$$

Properties of T':

- ord $T' = 24$
- $T'/A_4 \cong \mathbb{Z}_2 \Rightarrow$ irreps of A_4 are also irreps of T'
- T' has 7 classes $\Rightarrow$ 3 irreps missing
- $d_5^2 + d_6^2 + d_7^2 = 12 \Rightarrow d_5 = d_6 = d_7 = 2$
- $U_A^2 = U_R^3 = -\mathbb{1}$, $(U_A U_R)^3 = \mathbb{1}$

Eigenvalues of $U_A U_R$ are ω, ω^2

Finite groups used as family symmetries

2-dim irreps of T':
Idea: Suppose $D^{(\beta)}$ is an irrep with dimension $d > 1 \Rightarrow$
Two obvious ways for constructing further irreps of dimension d
Suppose $D^{(1,\alpha)}$ $(\alpha = 1, \ldots, r)$ lists all 1-dim irreps $\Rightarrow$
irreps of dim d are obtained by

1. $g \to D^{(1,\alpha)}(g) \times D^{(\beta)}(g)$
2. $g \to D^{(1,\alpha)}(g) \times (D^{(\beta)}(g))^*$

Method 1 works with $\mathbf{2}$ of T':

$$\begin{array}{lll} \mathbf{2}': & U_A \to U_A, & U_R \to \omega U_R \\ \mathbf{2}'': & U_A \to U_A, & U_R \to \omega^2 U_R \end{array}$$

Note:
T': $\mathbf{2} \cong \mathbf{2}^*$
A_4: $\mathbf{1}' \otimes \mathbf{3} \cong \mathbf{1}'' \otimes \mathbf{3} \cong \mathbf{3}\,!$

Functions on G:

Unitary space with scalar product $(f_1|f_2) = \frac{1}{\mathrm{ord}\, G} \sum_{g \in G} f_1^*(g) f_2(g)$

Theorem

$D^{(\alpha)}$ irreps of G with dimensions d_α $\Rightarrow$

$$\sum_{g \in G} D_{ij}^{(\alpha)}(g^{-1}) D_{kl}^{(\beta)}(g) = \frac{ordG}{d_\alpha} \delta_{\alpha\beta} \delta_{jk} \delta_{il}$$

Functions on G:

Unitary space with scalar product $(f_1|f_2) = \frac{1}{\operatorname{ord} G} \sum_{g \in G} f_1^*(g) f_2(g)$

Theorem

$D^{(\alpha)}$ irreps of G with dimensions d_α $\Rightarrow$

$$\sum_{g \in G} D^{(\alpha)}_{ij}(g^{-1}) D^{(\beta)}_{kl}(g) = \frac{\operatorname{ord} G}{d_\alpha} \delta_{\alpha\beta} \delta_{jk} \delta_{il}$$

Note:

Theorem follows from Schur's lemma

Unitary irrep $\Rightarrow D^{(\alpha)}_{ij}(g^{-1}) = (D^{(\alpha)}(g)^\dagger)_{ij} = (D^{(\alpha)}(g)_{ji})^*$

Schur's lemma:

1. D irrep on $\mathcal{V}$, A linear operator on $\mathcal{V}$ such that $AD = DA$
   $\Rightarrow A \propto \mathbb{1}$
2. Two non-equivalent irreps
   $D^{(1)}$ acting on $\mathcal{V}_1$, $D^{(2)}$ acting on $\mathcal{V}_2$,
   $A:\ \mathcal{V}_1 \to \mathcal{V}_2$ such that $AD^{(1)} = D^{(2)}A \Rightarrow A = 0$

Proof of theorem:

Part 1: D irrep on $\mathcal{V}$ with dimension d
$A \equiv \sum_{h \in G} D(h^{-1}) B D(h)$ with B arbitrary linear operator on $\mathcal{V}$
$\Rightarrow AD(g) = D(g)A \; \forall \, g \in G \Rightarrow A = \lambda \mathbb{1}$
Choose $B_{ij} = \delta_{ik}\delta_{jl}$
$A_{pq} = \lambda(kl)\delta_{pq} = \sum_{h \in G} D(h^{-1})_{pk} D(h)_{lq}$
Computation of $\lambda(kl)$ by summation over $p = q$:
$\lambda(kl)d = \sum_h \delta_{kl} = \mathrm{ord}\, G \, \delta_{kl} \Rightarrow$
$\sum_{h \in G} D(h^{-1})_{pk} D(h)_{lq} = (\mathrm{ord}\, G/d)\, \delta_{pq}\delta_{kl}$ ✔
Part 2: $B : \mathcal{V}_1 \to \mathcal{V}_2$ arbitrary $\Rightarrow A \equiv \sum_{h \in G} D^{(2)}(h^{-1}) B D^{(1)}(h)$
fulfills $AD^{(1)} = D^{(2)} A \Rightarrow A = 0$ etc. Q.E.D.

Character tables and tensor products

Definition

Character of an irrep:

$$\chi^{(\alpha)} : \begin{array}{ccc} G & \to & \mathbb{C} \\ g & \to & \chi^{(\alpha)}(g) = \operatorname{Tr} D^{(\alpha)}(g) \end{array}$$

Definition

Character of an irrep:

$$\chi^{(\alpha)} : \begin{array}{ccl} G & \to & \mathbb{C} \\ g & \to & \chi^{(\alpha)}(g) = \operatorname{Tr} D^{(\alpha)}(g) \end{array}$$

Properties of characters:

- The character $\chi^{(\alpha)}$ is constant on every class C_k $\Rightarrow$ denote value of character by $\chi_k^{(\alpha)}$ on C_k
- $\chi^{(\alpha)}(e) = d_\alpha$
- Let c_k be the number of elements in class C_k $\Rightarrow$ orthogonality relation

$$\sum_{k=1}^{n} c_k \left(\chi_k^{(\alpha)}\right)^* \chi_k^{(\beta)} = \delta_{\alpha\beta} \operatorname{ord} G$$

Character tables and tensor products

Proof of orthogonality relation:

$$\sum_{g \in G} (D^{(\alpha)}(g)_{ji})^* D^{(\beta)}_{kl}(g) = \frac{\mathrm{ord}\, G}{d_\alpha}\, \delta_{\alpha\beta}\delta_{jk}\delta_{il}$$

$i = j$, $k = l$, summation over i, k $\Rightarrow$

$$\sum_{g \in G} (\chi^{(\alpha)}(g))^* \chi^{(\beta)}(g) = \frac{\mathrm{ord}\, G}{d_\alpha}\, \delta_{\alpha\beta}\, d_\alpha = \mathrm{ord}\, G\, \delta_{\alpha\beta}$$

Sum over $g \in G$ $\rightsquigarrow$ sum over classes Q.E.D.

Proof of orthogonality relation:

$$\sum_{g\in G} (D^{(\alpha)}(g)_{ji})^* D^{(\beta)}_{kl}(g) = \frac{\operatorname{ord} G}{d_\alpha}\, \delta_{\alpha\beta}\delta_{jk}\delta_{il}$$

$i = j$, $k = l$, summation over i, k $\Rightarrow$

$$\sum_{g\in G} (\chi^{(\alpha)}(g))^* \chi^{(\beta)}(g) = \frac{\operatorname{ord} G}{d_\alpha}\, \delta_{\alpha\beta}\, d_\alpha = \operatorname{ord} G\, \delta_{\alpha\beta}$$

Sum over $g \in G$ $\rightsquigarrow$ sum over classes Q.E.D.

Remarks: $n = \#$ classes

ON system $\left(\sqrt{\frac{c_1}{\operatorname{ord} G}}\, \chi^{(\alpha)}_1, \ldots, \sqrt{\frac{c_n}{\operatorname{ord} G}}\, \chi^{(\alpha)}_n\right) \Rightarrow N_{\mathrm{irreps}} \leq n$

Proof that $N_{\mathrm{irreps}} \geq n$ needs different technique

Characters and reducible representations:

D reducible representation, $D^{(\alpha)}$ occurs n_α times in D
$\Rightarrow$ multiplicity of $D^{(\alpha)}$ in D given by $n_\alpha = (\chi^{(\alpha)}|\chi_D)$

Characters and reducible representations:

D reducible representation, $D^{(\alpha)}$ occurs n_α times in D
$\Rightarrow$ multiplicity of $D^{(\alpha)}$ in D given by $n_\alpha = (\chi^{(\alpha)}|\chi_D)$

Proof:

$$D(g) = \underbrace{D^{(\gamma)}(g) \oplus \cdots \oplus D^{(\gamma)}(g)}_{n_\gamma \text{ summands}} \oplus \underbrace{D^{(\delta)}(g) \oplus \cdots \oplus D^{(\delta)}(g)}_{n_\delta \text{ summands}} \oplus \cdots$$

$\Rightarrow \chi_D = \sum_\beta n_\beta \chi^{(\beta)}$ Q.E.D.

Application:
Characters allow e.g. to reduce $D^{(\beta)} \otimes D^{(\beta')}$
Motivation for character tables

Example: regular representation

Basis of vector space given by all group elements

$\{g_1, g_2, \ldots, g_N\}$ with $N \equiv \mathrm{ord}\, G$

$g_i g_j = R(g_i)_{kj} g_k \Rightarrow R(g_i)$ is $N \times N$ permutation matrix

$\chi_R(e) = N$, $\chi_R(g) = 0 \ \forall\, g \neq e$

$$
\begin{aligned}
(\chi^{(\alpha)}|\chi_R) &= \frac{1}{N} \sum_g (\chi^{(\alpha)}(g))^* \chi_R(g) \\
&= \frac{1}{N} (\chi^{(\alpha)}(e))^* \chi_R(e) \\
&= \frac{1}{N} \times d_\alpha \times N = d_\alpha
\end{aligned}
$$

Example: regular representation

Basis of vector space given by all group elements

$\{g_1, g_2, \ldots, g_N\}$ with $N \equiv \operatorname{ord} G$

$g_i g_j = R(g_i)_{kj} g_k \Rightarrow R(g_i)$ is $N \times N$ permutation matrix

$\chi_R(e) = N$, $\chi_R(g) = 0 \; \forall\, g \neq e$

$$
\begin{aligned}
(\chi^{(\alpha)}|\chi_R) &= \frac{1}{N} \sum_g (\chi^{(\alpha)}(g))^* \chi_R(g) \\
&= \frac{1}{N} (\chi^{(\alpha)}(e))^* \chi_R(e) \\
&= \frac{1}{N} \times d_\alpha \times N = d_\alpha
\end{aligned}
$$

$D^{(\alpha)}$ occurs d_α times in regular representation $\Rightarrow \sum_\alpha d_\alpha^2 = \operatorname{ord} G$

Character table:

n = number of irreps = number of classes
Classes C_k, c_k = number of members of C_k
$\mathrm{ord}\,(C_k)$ = order of elements of C_k
Irreps $D^{(\alpha)}$, characters $\chi^{(\alpha)}(g) = \mathrm{Tr}\left(D^{(\alpha)}(g)\right)$
$\chi_k^{(\alpha)} = \chi^{(\alpha)}(g)$ with $g \in C_k$

G $(\#\ C_k)$ $\mathrm{ord}\,(C_k)$	C_1 (c_1) ν_1	C_2 (c_2) ν_2	$\cdots$ $\cdots$ $\cdots$	C_n (c_n) ν_n
$D^{(1)}$	$\chi_1^{(1)}$	$\chi_2^{(1)}$	$\cdots$	$\chi_n^{(1)}$
$D^{(2)}$	$\chi_1^{(2)}$	$\chi_2^{(2)}$	$\cdots$	$\chi_n^{(2)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$D^{(n)}$	$\chi_1^{(n)}$	$\chi_2^{(n)}$	$\cdots$	$\chi_n^{(n)}$

Character tables and tensor products

Conventions: $D^{(1)}$ trivial 1-dim irrep $\Rightarrow \chi_k^{(1)} = 1 \; \forall k$

$C_1 = \{e\} \Rightarrow \chi_1^{(\alpha)} = d_\alpha$

First line and first column of character table

First line: 1 in all entries

First column: dimensions d_α of irreps

ON system $\sqrt{\frac{c_k}{\mathrm{ord}\, G}} \begin{pmatrix} \chi_k^{(1)} \\ \vdots \\ \chi_k^{(n)} \end{pmatrix}$ $(k = 1, \ldots, n)$

$$\sum_{\alpha=1}^{n} \left(\chi_k^{(\alpha)} \right)^* \chi_\ell^{(\alpha)} = \frac{\mathrm{ord}\, G}{c_k} \, \delta_{k\ell}$$

Character table of D_4:

Classes:

$C_1 = \{\mathbb{1}\}$, $C_2 = \{-\mathbb{1}\}$, $C_3 = \{\pm R_4\}$, $C_4 = \{\pm S\}$, $C_5 = \{\pm R_4 S\}$

1-dim irreps $\mathbf{1}^{(p,q)}$: $\quad S \to (-1)^p, \quad R_4 \to (-1)^q$

Note: $-\mathbb{1} = R_4^2 \Rightarrow -\mathbb{1} \to 1$ in 1-dim irreps

D_4 $(\#\ C_k)$ ord (C_k)	C_1 (1) 1	C_2 (1) 2	C_3 (2) 4	C_4 (2) 2	C_5 (2) 2
$\mathbf{1}^{(0,0)}$	1	1	1	1	1
$\mathbf{1}^{(1,0)}$	1	1	1	-1	-1
$\mathbf{1}^{(0,1)}$	1	1	-1	1	-1
$\mathbf{1}^{(1,1)}$	1	1	-1	-1	1
$\mathbf{2}$	2	-2	0	0	0

Example: $\mathbf{2} \otimes \mathbf{2}$

$\chi^{(\mathbf{2}\otimes\mathbf{2})} = \chi^{(\mathbf{2})} \times \chi^{(\mathbf{2})} = 4,\ 4,\ 0,\ 0,\ 0$

$(\chi^{(\mathbf{2})}|\chi^{(\mathbf{2}\otimes\mathbf{2})}) = \frac{1}{8}\,(2 \times 4 + (-2) \times 4) = 0$

$(\chi^{(\mathbf{1}^{(p,q)})}|\chi^{(\mathbf{2}\otimes\mathbf{2})}) = \frac{1}{8}\,(1 \times 4 + 1 \times 4) = 1$

$$\mathbf{2} \otimes \mathbf{2} = \mathbf{1}^{(0,0)} \oplus \mathbf{1}^{(1,0)} \oplus \mathbf{1}^{(0,1)} \oplus \mathbf{1}^{(1,1)}$$

Character table of A_4:

A_4 $(\#\ C_k)$ $\mathrm{ord}\,(C_k)$	C_1 (1) 1	C_2 (3) 2	C_3 (4) 3	C_4 (4) 3
$\mathbf{1}$	1	1	1	1
$\mathbf{1}'$	1	1	ω	ω^2
$\mathbf{1}''$	1	1	ω^2	ω
$\mathbf{3}$	3	-1	0	0

Example: $\chi^{(\mathbf{3}\otimes\mathbf{3})} = \chi^{(\mathbf{3})} \times \chi^{(\mathbf{3})} = 9,\ 1,\ 0,\ 0$

$(\chi^{(\mathbf{3})}|\chi^{(\mathbf{3}\otimes\mathbf{3})}) = \frac{1}{12}\,(3 \times 9 + 3 \times (-1) \times 1) = 2$

$(\chi^{(\mathbf{1})}|\chi^{(\mathbf{3}\otimes\mathbf{3})}) = (\chi^{(\mathbf{1}')}|\chi^{(\mathbf{3}\otimes\mathbf{3})}) = (\chi^{(\mathbf{1}'')}|\chi^{(\mathbf{3}\otimes\mathbf{3})}) = \frac{1}{12}\,(1 \times 9 + 3 \times 1 \times 1) = 1$

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{1}'' \oplus \mathbf{3} \oplus \mathbf{3}$$

Classes:
$C_1 = \{e\}$,
$C_2 = \{(12)(34),\ (13)(24),\ (14)(23)\}$,
$C_3 = \{(132),\ (124),\ (234),\ (143)\}$,
$C_4 = \{(123),\ (142),\ (243),\ (134)\}$

Generators of **3**:

$$(12)(34) \to A \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad (243) \to E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{ll} \mathbf{1}: & A \to 1,\ E \to 1 \\ \mathbf{1}': & A \to 1,\ E \to \omega^2 \\ \mathbf{1}'': & A \to 1,\ E \to \omega \end{array}$$

Clebsch–Gordan coefficients for $\mathbf{3} \otimes \mathbf{3}$ of A_4:

$$
\begin{aligned}
\mathbf{1}: &\quad e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3 \\
\mathbf{1}': &\quad e_1 \otimes e_1 + \omega^2 e_2 \otimes e_2 + \omega e_3 \otimes e_3 \\
\mathbf{1}'': &\quad e_1 \otimes e_1 + \omega e_2 \otimes e_2 + \omega^2 e_3 \otimes e_3 \\
\mathbf{3}: &\quad e_2 \otimes e_3,\ e_3 \otimes e_1,\ e_1 \otimes e_2; \\
&\quad e_3 \otimes e_2,\ e_1 \otimes e_3,\ e_2 \otimes e_1
\end{aligned}
$$

Character table of S_4:

Classes: $C_1 = \{e\}$, $C_2 =$ transposition (2-cycle), $C_3 =$ two transpositions, $C_4 =$ 3-cycle, $C_5 =$ 4-cycle

S_4 $(\# \; C_k)$ $\mathrm{ord}\,(C_k)$	C_1 (1) 1	C_2 (6) 2	C_3 (3) 2	C_4 (8) 3	C_5 (6) 4
$\mathbf{1}$	1	1	1	1	1
$\mathbf{1}'$	1	-1	1	1	-1
$\mathbf{2}$	2	0	2	-1	0
$\mathbf{3}$	3	1	-1	0	-1
$\mathbf{3}'$	3	-1	-1	0	1

$\chi^{(\mathbf{2})}$:

$$(12),\ (34) \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \chi_2^{(\mathbf{2})} = 0,\ \chi_3^{(\mathbf{2})} = 2$$

$$(234) \to \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix} \Rightarrow \chi_4^{(\mathbf{2})} = -1,$$

$$(1234) = (12)(234) \to \begin{pmatrix} 0 & \omega^2 \\ \omega & 0 \end{pmatrix} \Rightarrow \chi_5^{(\mathbf{2})} = 0$$

$\mathbf{3} \otimes \mathbf{3}$ of S_4:

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{2} \oplus \mathbf{3} \oplus \mathbf{3}'$$

$$\mathbf{1}: \quad e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3$$

$$\mathbf{2}: \quad \begin{cases} e_1 \otimes e_1 + \omega^2 e_2 \otimes e_2 + \omega e_3 \otimes e_3 \\ e_1 \otimes e_1 + \omega e_2 \otimes e_2 + \omega^2 e_3 \otimes e_3 \end{cases}$$

$$\mathbf{3}: \quad \begin{cases} \frac{1}{\sqrt{2}} (e_2 \otimes e_3 + e_3 \otimes e_2) \\ \frac{1}{\sqrt{2}} (e_3 \otimes e_1 + e_1 \otimes e_3) \\ \frac{1}{\sqrt{2}} (e_1 \otimes e_2 + e_2 \otimes e_1) \end{cases}$$

$$\mathbf{3}': \quad \begin{cases} \frac{1}{\sqrt{2}} (e_2 \otimes e_3 - e_3 \otimes e_2) \\ \frac{1}{\sqrt{2}} (e_3 \otimes e_1 - e_1 \otimes e_3) \\ \frac{1}{\sqrt{2}} (e_1 \otimes e_2 - e_2 \otimes e_1) \end{cases}$$

Symmetries in the neutrino mass matrix

- For the time being consider only lepton mass terms
- Assume diagonal charged-lepton mass matrix

Choose unitary 3×3 matrix S
Consider following symmetries in $\mathcal{L}_{\mathrm{Maj}}$:

1. Horizontal symmetry: $\boxed{\nu_L \to S\nu_L}$
2. Generalized CP symmetry: $\boxed{\nu_L \to iSC\nu_L^*}$

Symmetries in the neutrino mass matrix

- For the time being consider only lepton mass terms
- Assume diagonal charged-lepton mass matrix

Choose unitary 3×3 matrix S
Consider following symmetries in $\mathcal{L}_{\mathrm{Maj}}$:

1. Horizontal symmetry: $\boxed{\nu_L \to S\nu_L}$
2. Generalized CP symmetry: $\boxed{\nu_L \to iSC\nu_L^*}$

$$\boxed{\begin{array}{ll} \text{1} & \Rightarrow S^T \mathcal{M}_\nu S = \mathcal{M}_\nu \\ \text{2} & \Rightarrow S^T \mathcal{M}_\nu S = \mathcal{M}_\nu^* \end{array}}$$

Symmetries in the neutrino mass matrix

- For the time being consider only lepton mass terms
- Assume diagonal charged-lepton mass matrix

Choose unitary 3×3 matrix S
Consider following symmetries in $\mathcal{L}_{\mathrm{Maj}}$:

1. Horizontal symmetry: $\boxed{\nu_L \to S\nu_L}$
2. Generalized CP symmetry: $\boxed{\nu_L \to iSC\nu_L^*}$

$$\boxed{\begin{array}{ll} ① & \Rightarrow S^T \mathcal{M}_\nu S = \mathcal{M}_\nu \\ ② & \Rightarrow S^T \mathcal{M}_\nu S = \mathcal{M}_\nu^* \end{array}}$$

$\mathcal{M}_\nu = U^* \hat{m} U^\dagger$
① $\Rightarrow \hat{m} \left(U^\dagger S U \right) = \left(U^\dagger S U \right)^* \hat{m}$
② $\Rightarrow \hat{m} \left(U^\dagger S U^* \right) = \left(U^\dagger S U^* \right)^* \hat{m}$

Symmetries in the neutrino mass matrix

1. $W \equiv \left(U^\dagger S U\right)$ 2. $W \equiv \left(U^\dagger S U^*\right)$

$m_i W_{ij} = (W_{ij})^* m_j$ (no summation) $\Rightarrow m_i |W_{ij}| = |W_{ij}| m_j$
$\Rightarrow W_{ij} = 0$ for $i \neq j$
Assume for simplicity: $m_s \neq 0$

$$\boxed{\begin{array}{lllll} \text{1} & \Rightarrow U^\dagger S U = \hat{\epsilon} & \text{or} & S u_j = \epsilon_j u_j \\ \text{2} & \Rightarrow U^\dagger S U^* = \hat{\epsilon} & \text{or} & S u_j^* = \epsilon_j u_j \end{array}}$$

$\hat{\epsilon} = \mathrm{diag}\left(\epsilon_1, \epsilon_2, \epsilon_3\right)$ diagonal sign matrix

Symmetries in the neutrino mass matrix

① $W \equiv \left(U^\dagger S U\right)$ ② $W \equiv \left(U^\dagger S U^*\right)$

$m_i W_{ij} = (W_{ij})^* m_j$ (no summation) $\Rightarrow m_i |W_{ij}| = |W_{ij}| m_j$
$\Rightarrow W_{ij} = 0$ for $i \neq j$
Assume for simplicity: $m_s \neq 0$

$$\boxed{\begin{array}{llll} ① & \Rightarrow U^\dagger S U = \hat{\epsilon} & \text{or} & S u_j = \epsilon_j u_j \\ ② & \Rightarrow U^\dagger S U^* = \hat{\epsilon} & \text{or} & S u_j^* = \epsilon_j u_j \end{array}}$$

$\hat{\epsilon} = \mathrm{diag}\,(\epsilon_1, \epsilon_2, \epsilon_3)$ diagonal sign matrix

Choice of class of matrices S: "reflection"
$\mathbf{y} \in \mathbb{C}^3$ with $|y_1|^2 + |y_2|^2 + |y_3|^2 = 1$

$$\boxed{S_\mathbf{y} \equiv \mathbb{1} - 2\mathbf{y}\mathbf{y}^\dagger} \Rightarrow S_\mathbf{y}\mathbf{y} = -\mathbf{y}, \quad S_\mathbf{y}\mathbf{y}' = +\mathbf{y}' \text{ for } \mathbf{y}' \perp \mathbf{y}$$

Consequence for ①:

$\boxed{\text{Up to a phase factor, one of the } u_j \text{ in } U \text{ identical with } \mathbf{y}}$

μ–τ-symmetric neutrino mass matrix

$$\mathbf{y} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \sim u_3, \quad S_\mathbf{y} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\mathcal{M}_\nu = \begin{pmatrix} x & y & y \\ y & z & w \\ y & w & z \end{pmatrix}, \quad U = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -\frac{s_{12}}{\sqrt{2}} & \frac{c_{12}}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{s_{12}}{\sqrt{2}} & \frac{c_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} e^{i\hat{\beta}}$$

In general, $x, y, z, w \in \mathbb{C}$

6-parameter mass matrix $\mathcal{M}_\nu \Leftrightarrow m_{1,2,3}$, θ_{12}, $\beta_{1,2}$

Predictions: $\theta_{13} = 0^\circ$, $\theta_{23} = 45^\circ$

(CKM phase δ meaningless)

Trimaximal neutrino mass matrix

$$\mathbf{y} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \sim u_2, \quad S_{\mathbf{y}} = \frac{1}{3} \begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix}$$

$$\mathcal{M}_\nu = \begin{pmatrix} r+s & u & v \\ u & r+v & s \\ v & s & r+u \end{pmatrix} \quad \text{with} \quad r, s, u, v \in \mathbb{C}$$

7-parameter mass matrix $\mathcal{M}_\nu \Rightarrow$ 2 predictions

$$|U_{e2}|^2 = 1/3 \quad \Rightarrow \quad \boxed{s_{12}^2 = \frac{1}{3(1-s_{13}^2)} \geq \frac{1}{3}}$$

$$|U_{\mu 2}|^2 = 1/3 \quad \Rightarrow \quad \boxed{\tan 2\theta_{23} = \frac{1-2s_{13}^2}{s_{13} \cos\delta \sqrt{2-3s_{13}^2}}}$$

Tri-bimaximal lepton mixing

Harrison, Perkins, Scott (2002): TBM
Combine bimaximal and trimaximal $\Rightarrow$

$$U = \begin{pmatrix} 2/\sqrt{6} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix} e^{i\hat{\beta}} \equiv U_{\mathrm{HPS}}\, e^{i\hat{\beta}}$$

Bimaximal: $x + y = z + w$
trimaximal: $u = v$
5-parameter mass matrix $\mathcal{M}_\nu \Leftrightarrow m_{1,2,3},\ \beta_{1,2}$
Predictions: $\theta_{12} = 35.26^\circ$, $\theta_{23} = 45^\circ$, $\theta_{13} = 0^\circ$
(CKM phase δ meaningless)

Remark: Albright, Rodejohann (2008):
"trimaximal mixing" $u_1 \equiv (u_1)_{\mathrm{HPS}} \Rightarrow s_{12}^2 \leq 1/3$

CP-invariant neutrino mass matrix

Generalized CP transformation: $\nu_L \to i S_{\mathbf{y}} C \nu_L^*$

$$S_{\mathbf{y}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \Rightarrow \mathcal{M}_\nu = \begin{pmatrix} a & r & r^* \\ r & s & b \\ r^* & b & s^* \end{pmatrix} \quad r, s \in \mathbb{C}, \ \ a, b \in \mathbb{R}$$

Consequence for ❷ with above S:

$$\epsilon_j = 1 \ \Rightarrow \ u_j = \begin{pmatrix} c_j \\ d_j \\ d_j^* \end{pmatrix}, \quad \epsilon_j = -1 \ \Rightarrow \ u_j = \begin{pmatrix} i c_j \\ d_j \\ -d_j^* \end{pmatrix}$$

with $c_j \in \mathbb{R}, \ d_j \in \mathbb{C}$

$\Rightarrow \boxed{|U_{\mu j}| = |U_{\tau j}| \ \forall j = 1, 2, 3}$ Harrison, Scott (2002)

CP-invariant neutrino mass matrix

5-parameter mass matrix $\mathcal{M}_\nu \Leftrightarrow m_{1,2,3},\ \theta_{12},\ \theta_{13}$

Predictions:

$r^2 s^* \notin \mathbb{R} \Rightarrow \boxed{\theta_{23} = 45^\circ,\ e^{i\delta} = \pm i,\ e^{2i\beta_{1,2}} = \pm 1}$

$\sin^2 2\theta_{\mathrm{atm}} = 4\,|U_{\mu 3}|^2 \left(1 - |U_{\mu 3}|^2\right) = 1 - s_{13}^4$

Remark: $r^2 s^* \in \mathbb{R} \Leftrightarrow \sin\theta_{13} = 0 \rightarrow$
special case of μ–τ-symmetric $\mathcal{M}_\nu$

A 3-parameter neutrino mass matrix

Start with trimaximal mixing: $\omega \equiv e^{2\pi i/3}$

$$
\begin{aligned}
\mathcal{M}_\nu &= \begin{pmatrix} r+s & u & v \\ u & r+v & s \\ v & s & r+u \end{pmatrix} \\
&= \begin{pmatrix} x+y+t & z+\omega^2 y+\omega t & z+\omega y+\omega^2 t \\ z+\omega^2 y+\omega t & x+\omega y+\omega^2 t & z+y+t \\ z+\omega y+\omega^2 t & z+y+t & x+\omega^2 y+\omega t \end{pmatrix}
\end{aligned}
$$

Equivalent parameterizations! $(r = x - z)$

- Apply generalized μ–τ CP symmetry $\Rightarrow x, y, z, t \in \mathbb{R}$
- Assume $t = 0$

3-parameter neutrino mass matrix:

Grimus, Lavoura (2008)

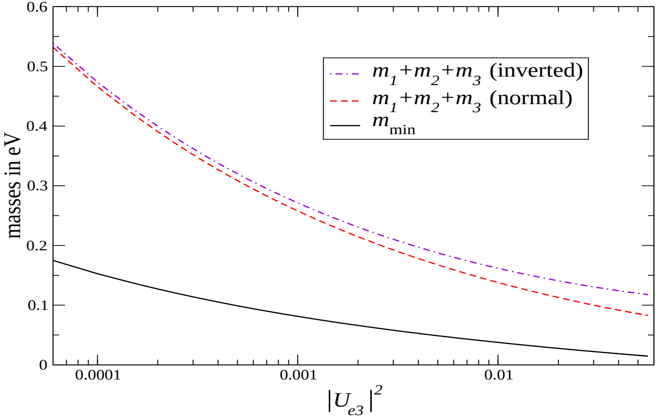
$$\mathcal{M}_\nu = \begin{pmatrix} x+y & z+\omega^2 y & z+\omega y \\ z+\omega^2 y & x+\omega y & z+y \\ z+\omega y & z+y & x+\omega^2 y \end{pmatrix}, \ \omega = e^{2\pi i/3}, \ x, y, z \in \mathbb{R}$$

3 parameters in $\mathcal{M}_\nu \Leftrightarrow \Delta m^2_{\mathrm{atm}}, \ \Delta m^2_\odot, \ \theta_{13}$

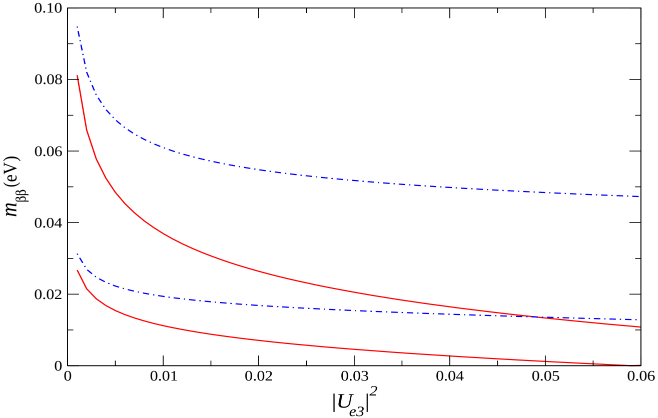
Predictions:

$$\boxed{\begin{gathered} s_{23} = \frac{1}{\sqrt{2}}, \ e^{i\delta} = \pm i, \ e^{2i\beta_{1,2}} = \pm 1 \\ s_{12}^2 = \frac{1}{3\left(1-s_{13}^2\right)} \\ m_s + \sqrt{m_s^2 + \Delta m^2_{\mathrm{atm}}} = \left[\frac{\left(\Delta m^2_{\mathrm{atm}}\right)^2}{3s_{13}^2\left(2-3s_{13}^2\right)}\right]^{1/4} \end{gathered}}$$

A 3-parameter neutrino mass matrix



$m_1 + m_2 + m_3$ and m_s as a function of $|U_{e3}|^2$
Neutrino mass-squared differences at their mean values



Neutrinoless $\beta\beta$ decay: $m_{\beta\beta}$ as a function of $|U_{e3}|^2$
Full lines: normal, dashed-dotted lines: inverted spectrum
Neutrino mass-squared differences at their mean values

Lagrangians and horizontal symmetries

- Horizontal symmetry $\equiv$ family symmetry: $G_{\mathrm{gauge}} \times G_{\mathrm{family}}$ (G_{family} could also be gauged)
- Kinetic and gauge terms in $\mathcal{L}$ invariant under G_{family}
- Effect of G_{family} in Yukawa Lagrangian and scalar potential
- Yukawa couplings connected with Clebsch–Gordan coefficients of tensor product of fermion representations
- Proliferation of scalar sector ($+$ additional fermion fields) vs. predictions for masses and mixings

Clebsch–Gordan coefficients vs. Yukawa couplings:

Tensor product: $D \otimes D' = D_S \oplus \cdots$ with irreps D, D', D_S
Bases: $D : \{e_\alpha\}$, $D' : \{f_\alpha\}$, $D_S : \{b_i = \Gamma_{i\alpha\beta} e_\alpha \otimes f_\beta\}$
Transformations:
$e_\alpha \to D_{\gamma\alpha} e_\gamma$, $f_\beta \to D_{\delta\alpha} f_\delta$, $b_i \to (D_S)_{ji} b_j$

Conditions on coefficient matrices Γ_i:

$$\Gamma_i = \left(D^\dagger \Gamma_j D'^* \right) (D_S)_{ji}$$

Generic Yukawa couplings: C = charge conjugation matrix

$$\mathcal{L}_Y = y \psi_\alpha^T C^{-1} \gamma_{i\alpha\beta} S_i \psi'_\beta + \text{H.c.}$$

$$\psi \to D\psi, \quad D' \to D'\psi' \quad \Rightarrow \quad S \to D_S^* S, \quad \gamma_i = \Gamma_i^*$$

Yukawa couplings partially determined by Clebsch–Gordan coefficients! $\Rightarrow$ reduction of number of parameters

Family symmetry groups in the Lagrangian:

- Can a symmetry in $\mathcal{M}_\nu$ be a remnant of a symmetry of a complete model of the lepton sector?
- What are the symmetries and multiplets of such a complete model?
- How can one achieve a charged-lepton mass matrix with freedom for adjusting $m_{e,\mu,\tau}$?

Fermion families 3-dimensional representations of horizontal group G:

1. Abelian: all fermion multiplets in $\mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{1}''$
2. non-Abelian: $\mathbf{1} \oplus \mathbf{2}$ occurs
3. non-Abelian: $\mathbf{3}$ occurs

Abelian group G:

Synonymous with "texture zeros" $\Rightarrow$
relations among observables

Grimus, Joshipura, Lavoura, Tanimoto (2004):

It is possible to enforce texture zeros in arbitrary entries of the fermion mass matrices by means of Abelian symmetries and an extended scalar sector

C. Low (2004):

Extremal mixing angles: only $\theta_{13} = 0^\circ$ can be enforced by Abelian symmetries

Non-Abelian symmetries:

$\mathbf{1} \oplus \mathbf{2}$: $G = D_n$ with $n \geq 3$ ($D_3 \cong S_3 \cong \Delta(6)$), $O(2)$, double-valued groups D'_n with $n \geq 2$ ($D'_2 \cong Q_8$)

Possible to enforce $\boxed{\theta_{13} = 0^\circ \text{ and } \theta_{23} = 45^\circ}$

Groups with 3-dim irreps: A_4, S_4, A_5
double-valued groups $A'_4 \equiv T'$, S'_4,
dihedral-like groups $\Delta(27)$, $\Delta(54)$
Lie groups $SO(3)$, $SU(3)$

- Relations among entries of mass matrices $\Rightarrow$ non-Abelian groups
- Clebsch–Gordan coefficients $\Rightarrow$ Yukawa couplings
- Avoiding extra Goldstone or gauge bosons from breaking of G $\Rightarrow$ finite groups
- Extended scalar sector $\Rightarrow$ problem of VEV alignment

Seesaw mechanism

Minkowski (1977)
Yanagida; Glashow; Gell-Mann, Ramond, Slansky (1979)
SM $+ 3\,\nu_R +$ total lepton number violation
Remark: could also choose 2 or more than 3 ν_R

$$\begin{aligned}\mathcal{L} &= \cdots - \textstyle\sum_j \left[\bar{\ell}_R \phi_j^\dagger \Gamma_j + \bar{\nu}_R \tilde{\phi}_j^\dagger \Delta_j\right] D_L + \text{H.c.} \\ &\quad + \left(\tfrac{1}{2}\,\nu_R^T C^{-1} M_R^* \nu_R + \text{h.c.}\right) \qquad \boxed{M_R = M_R^T}\end{aligned}$$

$$M_\ell = \frac{1}{\sqrt{2}} \sum_j v_j^* \Gamma_j\,, \quad M_D = \frac{1}{\sqrt{2}} \sum_j v_j \Delta_j$$

Total Majorana mass matrix for left-handed ν fields:

$$\boxed{\mathcal{M}_{D+M} = \begin{pmatrix} 0 & M_D^T \\ M_D & M_R \end{pmatrix} \quad \text{for} \quad \begin{pmatrix} \nu_L \\ C(\bar{\nu}_R)^T \end{pmatrix}}$$

Assumption: $m_D \ll m_R$ ($m_{D,R}$ scales of $M_{D,R}$)
$m_D \sim m_{e,\mu,\tau},\ m_Z$?

Seesaw mechanism:

Mass matrix of light neutrinos: $\boxed{\mathcal{M}_\nu = -M_D^T M_R^{-1} M_D}$

✤ $m_\nu \sim m_D^2 / m_R$ ✤

Mass matrix of heavy neutrinos: $\mathcal{M}_\nu^{\mathrm{heavy}} = M_R$

Diagonalization: $(U_R^\ell)^\dagger M_\ell U_L^\ell = \hat{m}_\ell,\ U_\nu^T \mathcal{M}_\nu U_\nu = \hat{m}$

Mixing matrix: $\boxed{U = (U_L^\ell)^\dagger U_\nu}$

A model for bimaximal mixing

Grimus, Lavoura (2001, 2002)

Consider following framework:

SM $+ \; n_H \, \phi + 3 \, \nu_R +$ soft $L_{e,\mu,\tau}$ breaking
Soft L_α breaking by ν_R mass term

Features of this framework:

- Seesaw mechanism
- Yukawa coupling matrices diagonal $\Rightarrow M_\ell$, M_D diagonal
- Soft breaking of lepton numbers at high scale
- M_R only source of ν mixing
- ν_R mass term has dim 3 $\Rightarrow$ soft $L_{e,\mu,\tau}$ breaking
  $\Rightarrow$ renormalizable models

A model for bimaximal mixing

Note: M_D, M_R μ–τ symmetric $\Leftrightarrow$ $\mathcal{M}_\nu$ μ–τ, i.e. invariant under $\nu_{\mu L} \leftrightarrow \nu_{\tau L}$ $\Leftrightarrow$

$$\mathcal{M}_\nu = \begin{pmatrix} x & y & y \\ y & z & w \\ y & w & z \end{pmatrix}$$

$$M_\ell \text{ diagonal} \quad \Rightarrow \quad U = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\frac{\sin\theta}{\sqrt{2}} & \frac{\cos\theta}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{\sin\theta}{\sqrt{2}} & \frac{\cos\theta}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$\theta \equiv \theta_{12}$ in general large but non-maximal, θ_{23} maximal, $U_{e3} = 0$

A model for bimaximal mixing

Realization of bimaximal mixing in soft-L-breaking framework:
SM with $3\,\phi + 3\,\nu_R$ + non-abelian horizontal symmetry group

$$\boxed{G_H \Leftarrow \left\{ \underbrace{U(1)_{L_e} \times U(1)_{L_\mu} \times U(1)_{L_\tau}}_{\text{softly broken}}\,,\; \underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2'}_{\text{spont. breaking}} \right\}}$$

$$\mathbb{Z}_2: \quad \left\{ \begin{array}{l} D_{\mu L} \leftrightarrow D_{\tau L},\; \nu_{\mu R} \leftrightarrow \nu_{\tau R},\; \mu_R \leftrightarrow \tau_R, \\ \phi_3 \to -\phi_3 \end{array} \right.$$

$$\mathbb{Z}_2': \quad e_R \to -e_R,\; \nu_{e,\mu,\tau R} \to -\nu_{e,\mu,\tau R},\; \phi_1 \to -\phi_1$$

Non-abelian "kernel" of this group is $O(2)$, could be replaced by D_n with $n \geq 3$

A model for bimaximal mixing

$$
\begin{aligned}
\mathcal{L}_Y \quad = \quad & -y_1 \bar{D}_{eL} \nu_{eR} \tilde{\phi}_1 - y_2 \left(\bar{D}_{\mu L} \nu_{\mu R} + \bar{D}_{\tau L} \nu_{\tau R} \right) \tilde{\phi}_1 \\
& -y_3 \bar{D}_{eL} e_R \phi_1 - y_4 \left(\bar{D}_{\mu L} \mu_R + \bar{D}_{\tau L} \tau_R \right) \phi_2 \\
& -y_5 \left(\bar{D}_{\mu L} \mu_R - \bar{D}_{\tau L} \tau_R \right) \phi_3 + \text{H.c.}
\end{aligned}
$$

$$
m_\mu = |y_4 v_2 + y_5 v_3| \,, \quad m_\tau = |y_4 v_2 - y_5 v_3|
$$

$$
\mathcal{L}_{\mathrm{Maj}} = \frac{1}{2} \nu_R^{-1} M_R^* C \bar{\nu}_R + \text{H.c.}
$$

Finetuning problem: $m_\mu \ll m_\tau$

A model for bimaximal mixing

How the model functions:

- $\hookrightarrow$ μ–τ symmetry $\mathbb{Z}_2 \Rightarrow M_{D\mu\mu} = M_{D\tau\tau}$, M_R, $\mathcal{M}_\nu$ μ–τ symmetric
- $\hookrightarrow$ M_ℓ diagonal because of lepton numbers
- $\hookrightarrow$ Auxiliary symmetry $\mathbb{Z}_2' \Rightarrow \phi_{2,3}$ do not couple to $\bar{D}_L \nu_R$
- $\hookrightarrow$ $\mathbb{Z}_2$ spontanously broken VEV of $\phi_3 \Rightarrow m_\mu \neq m_\tau$
- $\hookrightarrow$ $\Delta m^2_\odot / \Delta m^2_{\mathrm{atm}} \sim 1/30$ reproduced by by tuning

Non-decoupling in the scalar sector (neutral scalar vertices) for $m_R \to \infty \Rightarrow$

- $\triangleright$ Amplitudes of $\mu \to e\gamma$, $Z \to e^- \mu^+, \ldots \propto 1/m_R^2$
- $\triangleright$ Amplitude of, e.g., $\mu \to 3e$ constant, suppressed by product of 4 Yukawa couplings, within exp. reach?

A type I seesaw model with A_4

A_4 is smallest finite group (12 elements) with 3-dim irrep
Ma, Rajasekaran (2001) for lepton sector
(in quark sector Wyler (1979), Branco, Nilles, Rittenberg (1980))
Generators of **3**:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\mathbf{1}': \quad E \to \omega^2, \quad A \to 1, \quad \mathbf{1}'': \quad E \to \omega, \quad A \to 1 \quad (\omega = e^{2\pi i/3})$$

$$\boxed{\mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{1}'' \oplus \mathbf{3} \oplus \mathbf{3}}$$

$$\begin{array}{rc} \mathbf{1}: & e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3 \\ \mathbf{1}': & e_1 \otimes e_1 + \omega^2 e_2 \otimes e_2 + \omega e_3 \otimes e_3 \\ \mathbf{1}'': & e_1 \otimes e_1 + \omega e_2 \otimes e_2 + \omega^2 e_3 \otimes e_3 \\ \mathbf{3}: & e_2 \otimes e_3,\ e_3 \otimes e_1,\ e_1 \otimes e_2; \\ & e_3 \otimes e_2,\ e_1 \otimes e_3,\ e_2 \otimes e_1 \end{array}$$

A type I seesaw model with A_4

He, Keum, Volkas (2005)

fermion fields: $\ell_R \in \mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{1}''$, $D_L,\ \nu_R \in \mathbf{3}$
scalar fields: doublets $\phi \in \mathbf{3}$, $\phi_0 \in \mathbf{1}$, real singlets $\chi \in \mathbf{3}$

$$
\begin{aligned}
\mathcal{L}_Y \quad = \quad & \cdots - \Big[h_1 \left(\bar{D}_{1L}\phi_1 + \bar{D}_{2L}\phi_2 + \bar{D}_{3L}\phi_3 \right) \ell_{1R} \\
& \qquad + h_2 \left(\bar{D}_{1L}\phi_1 + \omega \bar{D}_{2L}\phi_2 + \omega^2 \bar{D}_{3L}\phi_3 \right) \ell_{2R} \\
& \qquad + h_3 \left(\bar{D}_{1L}\phi_1 + \omega^2 \bar{D}_{2L}\phi_2 + \omega \bar{D}_{3L}\phi_3 \right) \ell_{3R} \\
& \qquad + h_0 \left(\bar{D}_{1L}\nu_{1R} + \bar{D}_{2L}\nu_{2R} + \bar{D}_{3L}\nu_{3R} \right) \tilde{\phi}_0 + \text{H.c.} \Big] \\
& + \frac{1}{2} \Big[M \left(\nu_{1R}^T C^{-1} \nu_{1R} + \nu_{2R}^T C^{-1} \nu_{2R} + \nu_{3R}^T C^{-1} \nu_{3R} \right) + \text{H.c.} \Big] \\
& + \frac{1}{2} \Big[h_\chi \Big(\chi_1 \left(\nu_{2R}^T C^{-1} \nu_{3R} + \nu_{3R}^T C^{-1} \nu_{2R} \right) \\
& \qquad\quad + \chi_2 \left(\nu_{3R}^T C^{-1} \nu_{1R} + \nu_{1R}^T C^{-1} \nu_{3R} \right) \\
& \qquad\quad + \chi_3 \left(\nu_{1R}^T C^{-1} \nu_{2R} + \nu_{3R}^T C^{-1} \nu_{1R} \right) \Big) + \text{H.c.} \Big]
\end{aligned}
$$

A type I seesaw model with A_4

Mass matrices: $M_D = h_0 v_0 \mathbb{1}$,

$$M_\ell = \begin{pmatrix} h_1 v_1^* & h_1 v_2^* & h_1 v_3^* \\ h_2 v_1^* & h_2 v_2^* \omega^2 & h_2 v_3^* \omega \\ h_3 v_1^* & h_3 v_2^* \omega & h_3 v_3^* \omega^2 \end{pmatrix}, \quad M_R = \begin{pmatrix} M & h_\chi w_3 & h_\chi w_2 \\ h_\chi w_3 & M & h_\chi w_1 \\ h_\chi w_2 & h_\chi w_1 & M \end{pmatrix}$$

Vacuum alignment:

$$v_1 = v_2 = v_3 \equiv v \;\Rightarrow\; M_\ell = \sqrt{3} v \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix} U_\omega^\dagger$$

$$w_1 = w_3 = 0,\; h_\chi w_2 \equiv M' \;\Rightarrow\; M_R = \begin{pmatrix} M & 0 & M' \\ 0 & M & 0 \\ M' & 0 & M \end{pmatrix}$$

A type I seesaw model with A_4

$$U_L^\ell \equiv U_\omega = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \quad U_\nu = \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}$$

$$U = (U_L^\ell)^\dagger U_\nu = U_\omega^\dagger U_\nu = \mathrm{diag}\,(1,\, \omega^2,\, \omega)\, U_{\mathrm{HPS}}\, \mathrm{diag}\,(1,\, 1,\, i)$$

Comments:

- Tri-bimaximal mixing for the vacuum alignment shown on previous slide
- Vacuum alignment: possible if scalar potential is CP-conserving
- ℓ_R not in the same A_4 multiplet as D_L, $\nu_R \Rightarrow$ cannot be embedded into a GUT

Model with $D_L, \ell_R \in \mathbf{3}$, type II seesaw:

Altarelli, Fergulio (2005), Ma (2006)

Scalar sector:

gauge doublets $(\phi_0,\ \phi_1,\ \phi_2,\ \phi_3) \in \mathbf{1} \oplus \mathbf{3}$

gauge triplets $(\xi_0,\ \xi_1,\ \xi_2,\ \xi_3) \in \mathbf{1} \oplus \mathbf{3}$

$$
\begin{aligned}
M_\ell &= \begin{pmatrix} h_0 v_0 & h_1 v_3 & h_2 v_2 \\ h_2 v_3 & h_0 v_0 & h_1 v_1 \\ h_1 v_2 & h_2 v_1 & h_0 v_0 \end{pmatrix} \\
M_\nu &= \begin{pmatrix} a & d & c \\ d & a & b \\ c & b & a \end{pmatrix}
\end{aligned}
$$

$\langle \phi_j^0 \rangle \equiv v_j$, $\langle \xi_k^0 \rangle \propto a, \ldots, d$

VEV alignment for TBM: $\boxed{v_1 = v_2 = v_3,\ c = d = 0}$

$$U_\ell \equiv U_\omega = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \quad \text{with} \quad U_\ell^\dagger M_\ell U_\ell \text{ diagonal}$$

$$\begin{aligned} m_e &= |h_0 v_0 + (h_1 + h_2) v| \\ m_\mu &= |h_0 v_0 + (h_1 \omega + h_2 \omega^2) v| \\ m_\tau &= |h_0 v_0 + (h_1 \omega^2 + h_2 \omega) v| \end{aligned}$$

$$U_\nu = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\boxed{U_\ell^\dagger U_\nu = U_{\mathrm{HPS}} \, \mathrm{diag}\,(1, 1, -i)}$$

Some comments on A_4 models

Remarks:

- VEV alignment and parameter finetuning
  Altarelli, Feruglio, Hagedorn, . . .
  non-renormalizable terms, SUSY (new physics at TeV scale), extra dimensions, . . .
- Breaking of family group G to different subgroups in charged lepton and neutrino sectors $\Rightarrow$ TBM
  Blum, Hagedorn, Lindner; Altarelli, Feruglio; . . .
  Charged lepton sector: $A_4 \to \mathbb{Z}_3(E)$ (cyclic permutations)
  Neutrino sector: $A_4 \to \mathbb{Z}_2(A)$,
  additional $\mathbb{Z}_2$ through $2 \leftrightarrow 3$ symmetry
- Is S_4 *the* symmetry group for TBM? (Lam (2008))
  Statement: $S_4 \subseteq G$

Trimaximal mixing with 4 right-handed neutrino singlets

Construction of seesaw models with more than three ν_R
Grimus, Lavoura (2008)

- Use more than three ν_R (4 or 5) for model building
- Enforce diagonal charged-lepton mass matrix
- Discussion of two cases:
    1. Trimaximal mixing in $\mathcal{M}_\nu$ with 3 parameters (four ν_R)
    2. TBM mixing (five ν_R)

Trimaximal mixing with 4 right-handed neutrino singlets

Charged-lepton sector:

Fermionic gauge multiplets:
doublets $D_{\alpha L}$, singlets α_R and $\nu_{\alpha R}$ $(\alpha = e, \mu, \tau)$

$$\boxed{\text{Family lepton number symmetries } U_{L_\alpha}}$$

$\Rightarrow$ fermion bilinears $\bar{D}_{\alpha L} \alpha_R$
Simplest way to achieve different masses $m_{e,\mu,\tau}$:
introduce Higgs doublet ϕ_α for each flavour

$$y_1 \sum_{\alpha=e,\mu,\tau} \bar{D}_{\alpha L} \alpha_R \phi_\alpha \;\Rightarrow\; m_\alpha = |y_1 v_\alpha|$$

Different charged-lepton masses by different VEVs!
Further symmetries of Yukawa couplings:

$$\boxed{\text{Flavour permutation symmetry } S_3}$$

$$\boxed{\text{“Higgs doublet numbers” } \mathbf{z}_\alpha : \alpha_R \to -\alpha_R,\ \phi_\alpha \to -\phi_\alpha}$$

Yukawa couplings of neutrino singlets:

Add Higgs doublet $\phi_0 \to$ flavour singlet

Lepton family symmetries + permutation symmetry S_3:

$$y_2 \sum_{\alpha=e,\mu,\tau} \bar{D}_{\alpha L} \nu_{\alpha R} \tilde{\phi}_0 \quad \text{with} \quad \tilde{\phi}_0 = i\tau_2 \phi_0^*$$

Breaking of family symmetries:

In ν_R mass term $\Rightarrow$ soft breaking by dimension 3

S_3-invariant mass term:

$$\frac{1}{2} M_0^* \sum_\alpha \nu_{\alpha R}^T C^{-1} \nu_{\alpha R} +$$

$$M_1^* \left(\nu_{eR}^T C^{-1} \nu_{\mu R} + \nu_{\mu R}^T C^{-1} \nu_{\tau R} + \nu_{\tau R}^T C^{-1} \nu_{eR} \right)$$

Trimaximal mixing with 4 right-handed neutrino singlets

Seesaw mechanism:

$$\begin{aligned}\mathcal{L}_{\nu\,\mathrm{mass}} &= \left(-\bar{\nu}_R M_D \nu_L + \tfrac{1}{2}\nu_R^T C^{-1} M_R \nu_R\right) + \text{H.c.} \\ &= \tfrac{1}{2}\,\omega_L^T C^{-1} \mathcal{M}_{D+M}\,\omega_L + \text{H.c.}\end{aligned}$$

$$\mathcal{M}_{D+M} = \begin{pmatrix} 0 & M_D^T \\ M_D & M_R \end{pmatrix} \quad \text{with} \quad \omega_L = \begin{pmatrix} \nu_L \\ C(\bar{\nu}_R)^T \end{pmatrix}$$

Mass matrix of light neutrinos: $\boxed{\mathcal{M}_\nu = -M_D^T M_R^{-1} M_D}$

Application to model construction:

$$M_R = \begin{pmatrix} M_0 & M_1 & M_1 \\ M_1 & M_0 & M_1 \\ M_1 & M_1 & M_0 \end{pmatrix},\ M_D \propto \mathbb{1}_3 \Rightarrow \begin{cases} U = U_{\mathrm{HPS}} & \text{fine!} \\ m_1 = m_3 & \text{failure!} \end{cases}$$

Trimaximal mixing with 4 right-handed neutrino singlets

Addition of one right-handed neutrino singlet:

- Reduce S_3 to cyclic permutations $(\mathbb{1}, \mathcal{C}, \mathcal{C}^2)$
- Additional ν_R singlet: $\mathcal{C}: \ \nu_{0R} \to \omega \nu_{0R}$
- Complex scalar singlet: $\mathcal{C}: \ \chi \to \omega \chi$

$$\left[\frac{1}{2} y_\chi \, \nu_{0R}^T C^{-1} \nu_{0R} \, \chi + \frac{1}{2} M_2^* \left(\nu_{eR}^T + \omega \, \nu_{\mu R}^T + \omega^2 \, \nu_{\tau R}^T \right) C^{-1} \nu_{0R} \right] + \text{H.c.}$$

Trimaximal mixing with 4 right-handed neutrino singlets

$M_N \equiv y_\chi v_\chi^*$, $a \equiv y_2^* v_0$

$$M_R = \begin{pmatrix} M_0 & M_1 & M_1 & M_2 \\ M_1 & M_0 & M_1 & \omega^2 M_2 \\ M_1 & M_1 & M_0 & \omega M_2 \\ M_2 & \omega^2 M_2 & \omega M_2 & M_N \end{pmatrix} \quad M_D = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix},$$

Seesaw formula $\Rightarrow$

$$\boxed{\mathcal{M}_\nu = \begin{pmatrix} x+y & z+\omega^2 y & z+\omega y \\ z+\omega^2 y & x+\omega y & z+y \\ z+\omega y & z+y & x+\omega^2 y \end{pmatrix}}$$

Trimaximal mixing with 4 right-handed neutrino singlets

$$
\begin{aligned}
x &= -a^2 \frac{M_0 + M_1}{(M_0 - M_1)(M_0 + 2M_1)} \\
z &= a^2 \frac{M_1}{(M_0 - M_1)(M_0 + 2M_1)} \\
y &= -a^2 \frac{M_2^2}{M_N (M_0 - M_1)^2}
\end{aligned}
$$

Seesaw mechanism: $M_{0,1,2,N}$ of large seesaw scale
$\Rightarrow v_\chi$ of seesaw scale

Trimaximal mixing with 4 right-handed neutrino singlets

Addition of generalized CP symmetry:

$$S_{\mathbf{y}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$D_L \to iS_{\mathbf{y}} C D_L^*, \quad \ell_R \to iS_{\mathbf{y}} C \ell_R^*, \quad \nu_R \to iS_{\mathbf{y}} C \nu_R^*, \quad \nu_{0R} \to iC\nu_{0R}^*$$
$$\phi \to S_{\mathbf{y}} \phi^*, \quad \phi_0 \to \phi_0^*, \quad \chi \to \chi^*$$

CP transformation $\Rightarrow y_1, y_2, y_\chi, M_0, M_1, M_2 \in \mathbb{R}$
Trivial condition on scalar potential: v_χ real $\Rightarrow M_N \in \mathbb{R}$
$\Rightarrow x, y, z \in \mathbb{R} \Rightarrow \mathcal{M}_\nu$ trimaximal 3-parameter mass matrix

Symmetry breaking summary:

dim of terms in $\mathcal{L}$	conserved symmetries
dim 4	CP, $\mathbf{z}_\alpha$, $\mathcal{C}$, U_{L_α}
dim 3	CP, $\mathbf{z}_\alpha$, $\mathcal{C}$
dim 2	CP

Remarks:
Eventually, all symmetries are broken spontaneously.
Spontaneous CP breaking $\Leftrightarrow$ $m_\mu \neq m_\tau$

Trimaximal mixing with 4 right-handed neutrino singlets

Model summary:

- Extension of the SM with 4 ν_R, 4 Higgs doublets, one scalar singlet
- Minimal number of Yukawa couplings
- $m_e : m_\mu : m_\tau = |v_e| : |v_\mu| : |v_\tau|$
- Lepton mixing solely from ν_R mass matrix M_R
- Chain of soft symmetry breaking

$$G = [(U(1) \times U(1) \times U(1)) \times (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)] \rtimes \mathbb{Z}_3$$

TBM with 5 right-handed neutrino singlets

Two additional neutrino singlets:

- Modification of previous model: Addition of neutrino singlets ν_{1R}, ν_{2R} and complex scalar singlet χ
- Keep full S_3 in terms of dim 4 and 3 in $\mathcal{L}$:
  - $\mathcal{C}$: $\nu_{1R} \to \omega\nu_{1R}$, $\nu_{2R} \to \omega^2\nu_{2R}$, $\chi \to \omega\chi$
  - $I_{\mu\tau}$: $\nu_{1R} \leftrightarrow \nu_{2R}$, $\chi \leftrightarrow \chi^*$
- $I_{\mu\tau}$ *spontaneously* broken with *real* VEV v_χ (trivial condition in the scalar potential)

Note:

$$\text{2-dim irrep} \begin{pmatrix} \chi \\ \chi^* \end{pmatrix} \text{ of } S_3 \cong D_3 : \quad \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

TBM with 5 right-handed neutrino singlets

$$\mathcal{L} = \cdots + \frac{1}{2} y_3 \left(\chi\, \nu_{1R}^T C^{-1} \nu_{1R} + \chi^*\, \nu_{2R}^T C^{-1} \nu_{2R} \right)$$
$$+ M_2^* \left[\nu_{1R}^T C^{-1} \left(\nu_{eR} + \omega \nu_{\mu R} + \omega^2 \nu_{\tau R} \right) \right.$$
$$\left. + \nu_{2R}^T C^{-1} \left(\nu_{eR} + \omega^2 \nu_{\mu R} + \omega \nu_{\tau R} \right) \right]$$
$$+ M_4^* \nu_{1R}^T C^{-1} \nu_{2R} + \cdots$$

$$M_R = \begin{pmatrix} M_0 & M_1 & M_1 & M_2 & M_2 \\ M_1 & M_0 & M_1 & \omega^2 M_2 & \omega M_2 \\ M_1 & M_1 & M_0 & \omega M_2 & \omega^2 M_2 \\ M_2 & \omega^2 M_2 & \omega M_2 & M_N & M_4 \\ M_2 & \omega M_2 & \omega^2 M_2 & M_4 & M_N' \end{pmatrix}, \quad M_D = \begin{pmatrix} a \mathbb{1}_{3\times 3} \\ 0_{2\times 3} \end{pmatrix}$$

with $M_N \equiv y_3^* v_\chi^*$, $M_N' \equiv y_3^* v_\chi$

Seesaw mechanism: $\mathcal{M}_\nu = -M_D^T M_R^{-1} M_D \Rightarrow$

$$\mathcal{M}_\nu = \begin{pmatrix} x+y+t & z+\omega^2 y+\omega t & z+\omega y+\omega^2 t \\ z+\omega^2 y+\omega t & x+\omega y+\omega^2 t & z+y+t \\ z+\omega y+\omega^2 t & z+y+t & x+\omega^2 y+\omega t \end{pmatrix}$$

$y/t = v_\chi / v_\chi^* \Rightarrow [v_\chi \text{ real} \Leftrightarrow y = t]$ and

$$\mathcal{M}_\nu = \begin{pmatrix} x+2y & z-y & z-y \\ z-y & x-y & z+2y \\ z-y & z+2y & x-y \end{pmatrix} \Rightarrow \text{TBM}$$

$$G = [(U(1)\times U(1)\times U(1))\times(\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2)] \rtimes S_3$$

Conclusions

- Symmetries based on finite groups could be a way to tackle the mass and mixing problem.
- Models for lepton mixing (and neutrino masses?) require complicated/contrived extensions of SM
- Such models are in most cases incompatible with Grand Unification
- A route for such models, avoiding vacuum alignment, SUSY, non-renormalizable terms, . . . , could be an enlarged ν_R sector (plus extended scalar sector)
-
  - For the time being, bimaximal and tri-bimaximal mixing are compatible with all experimental results.
  - However, if $s^2_{13} \sim 0.01$, then alternative ideas are needed.
  - Or a degenerate ν-mass spectrum $\Rightarrow s^2_{13} \neq 0$ by RGE from high (seesaw) scale to ew. scale.