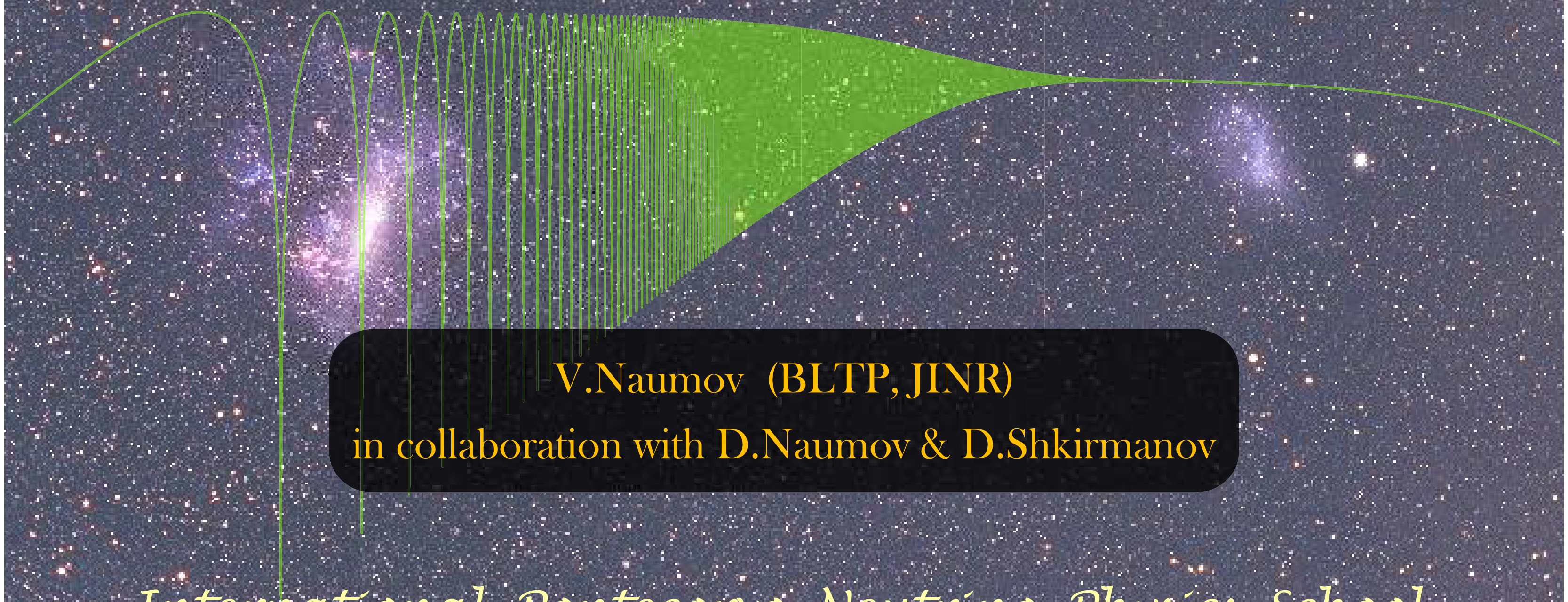
NEUTRINO OSCILLATIONS
A COVARIANT FIELD-THEORETICAL APPROACH



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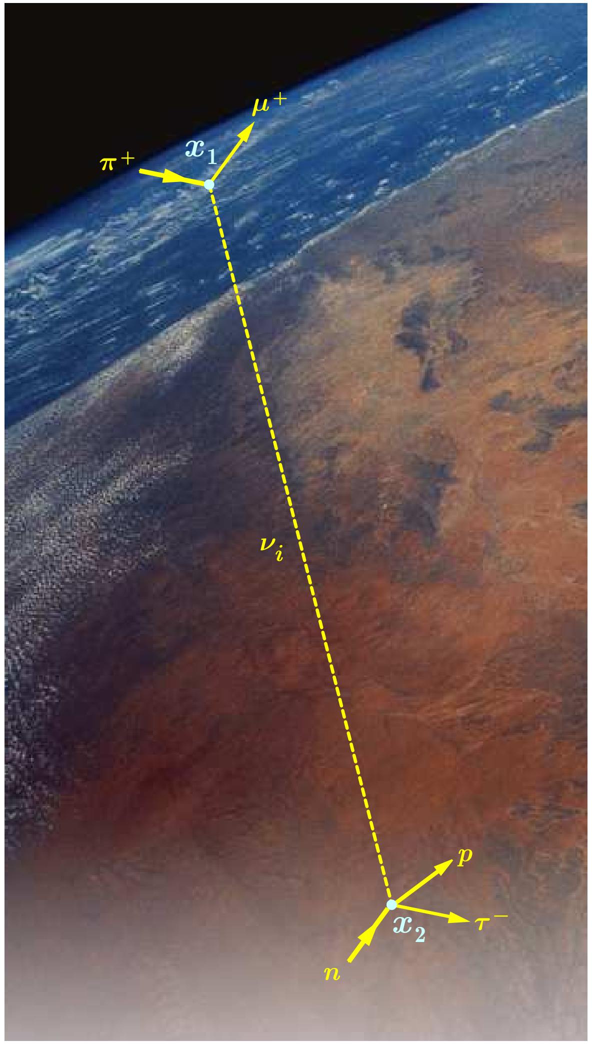
The aims and concepts of the field-theoretical approach.

The main purposes:

To define the domain of applicability of the standard quantum-mechanical (QM) theory of vacuum neutrino oscillations and obtain the QFT corrections to it.

The basic concepts:

- The "ν-oscillation" phenomenon in QFT is nothing else than a result of interference of the macroscopic Feynman diagrams perturbatively describing the lepton number violating processes with the massive neutrino fields as internal lines (propagators).
- The external lines of the macrodiagrams are wave packets rather than plane waves (therefore the standard S matrix approach should be revised).
- The external wave packet states are the covariant superpositions of the standard one-particle Fock states, satisfying a correspondence principle.



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The list is probably incomplete, apologies for any omissions.

Angels & hippopotami

According to the current theoretical understanding, the neutrino fields/states of definite flavor are superpositions of the fields/states with definite, generally different masses [and vice versa]:

$$\nu_\alpha = \sum_i V_{\alpha i}\nu_i \qquad \text{for neutrino fields,}$$

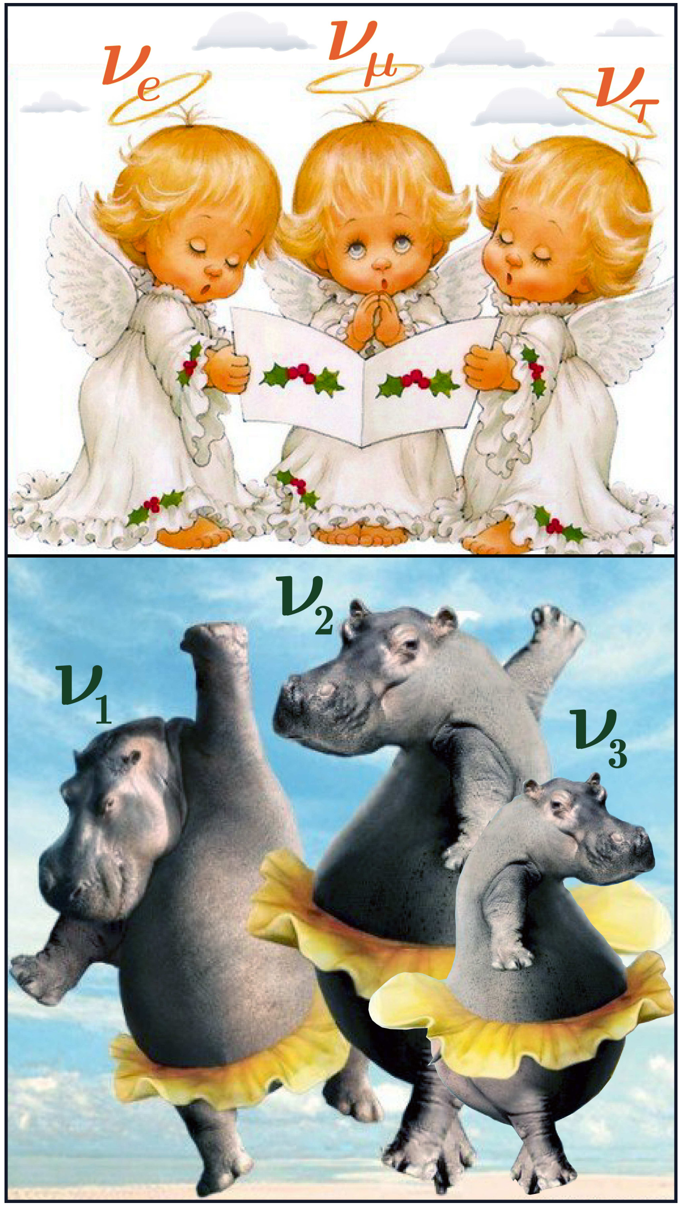
$$|\nu_\alpha\rangle = \sum_i V^*_{\alpha i}|\nu_i\rangle \qquad \text{for neutrino states;}$$

$$\alpha = e, \mu, \tau, \quad i = 1, 2, 3, \ldots$$

Here $V_{\alpha i}$ are the elements of the Pontecorvo-Maki-Nakagawa-Sakata neutrino vacuum mixing matrix $\mathbf{V}$.

This concept leads to the possibility of transitions between different flavor neutrinos, $\nu_\alpha \longleftrightarrow \nu_\beta$, phenomenon known as neutrino flavor oscillations.

We'll not appeal to the sweet but a bit ephemeral cherubs (neutrino flavor eigenfields/eigenstates) and will only deal with the more prosaic hippos (neutrino mass eigenfields/eigenstates).



Some challenges against the QM approach

Equal-momentum assumption

Massive neutrinos ν_i have, by assumption, equal momenta: $\mathbf{p}_i = \mathbf{p}_\nu$ $(i = 1, 2, 3)$.
This key assumption seems to be unphysical being reference-frame (RF) dependent;
if it is true in a certain RF then it is false in another RF moving with the velocity $\mathbf{v}$:

$$E_i' = \Gamma_{\mathbf{v}} \left[E_i - (\mathbf{v}\mathbf{p}_\nu) \right], \quad \mathbf{p}_i' = \mathbf{p}_\nu + \Gamma_{\mathbf{v}} \left[\frac{\Gamma_{\mathbf{v}}(\mathbf{v}\mathbf{p}_\nu)}{\Gamma_{\mathbf{v}} + 1} - E_i \right] \mathbf{v},$$

$\Downarrow$ [assuming, as necessary for oscillations, that $m_i \neq m_j$] $\Downarrow$

$$\mathbf{p}_i' - \mathbf{p}_j' = \left(E_j' - E_i' \right) \mathbf{v} = \Gamma_{\mathbf{v}} \left(E_j - E_i \right) \mathbf{v} \neq 0.$$

Treating the Lorentz transformation as active, we conclude that the EM assumption cannot be applied to the non-monoenergetic ν beams (the case in real-life experiments).

∗ A similar objection exists against the alternative equal-energy assumption; in that case

$$E_i' - E_j' = \Gamma_{\mathbf{v}} \left(\mathbf{p}_j - \mathbf{p}_i \right) \mathbf{v} \neq 0, \quad \left| \mathbf{p}_i' - \mathbf{p}_j' \right| = \sqrt{\left| \mathbf{p}_i - \mathbf{p}_j \right|^2 + \Gamma_{\mathbf{v}}^2 \left[\left(\mathbf{p}_i - \mathbf{p}_j \right) \mathbf{v} \right]^2} \neq 0.$$

∗ Can the EM (or EE) assumption be at least a good approximation? Alas, no, it cannot.

> Let ν_μs arise from $\pi_{\mu 2}$ decays. If the pion beam has a wide momentum spectrum – from subrelativistic to ultrarelativistic (as it is, e.g., for cosmic-ray particles), the EM (or EE) condition cannot be valid even approximately within the whole spectral range of the pion neutrinos.

Light-ray approximation

The propagation time T is, by assumption, equal to the distance L traveled by the neutrino between production and detection points. But, if the massive neutrino components have the same momentum $\mathbf{p}_\nu$, their velocities are in fact different:

$$\mathbf{v}_i = \frac{\mathbf{p}_\nu}{\sqrt{\mathbf{p}_\nu^2 + m_i^2}} \quad \Longrightarrow \quad |\mathbf{v}_i - \mathbf{v}_j| \approx \frac{\Delta m_{ji}^2}{2E_\nu^2}.$$

One may naively expect that during the time T the neutrino ν_i travels the distance $L_i = |\mathbf{v}_i|T$; therefore, there must be a spread in distances of each neutrino pair

$$\delta L_{ij} = L_i - L_j \approx \frac{\Delta m_{ji}^2}{2E_\nu^2} L, \quad \text{where} \quad L = cT = T.$$

Δm_{ji}^2	E_ν	L	L_{ij}	$\lvert\delta L_{ij}\rvert$
Δm_{23}^2	1 GeV	$2R_\oplus$	$0.1R_\oplus$	$\sim 10^{-12}$ cm
Δm_{23}^2	1 TeV	$R_G \sim 100$ kps	$100R_\oplus$	$\sim 10^{-4}$ cm
Δm_{21}^2	1 MeV	1 AU	$0.25R_\oplus$	$\sim 10^{-3}$ cm

The values of δL_{ij} listed in the Table seem to be **fantastically** small. But

Are they sufficiently small to preserve the coherence in any circumstance?

In other words:

What is the natural scale of the distances and times?

Can light neutrinos oscillate into heavy ones or vise versa?
[Can active neutrinos oscillate into sterile ones or vise versa?]

The naive QM answer is Yes. Why not? If, at least, both ν_α (light) and ν_s (heavy) are ultrarelativistic [$|\mathbf{p}_\nu| \gg \max(m_1, m_2, m_3, \ldots, M)$,] one obtains the same formula for the oscillation probability $\mathscr{P}_{\alpha s}(L)$, since the QM formalism has no any limitation to the neutrino mass hierarchy.

Possibility of such transitions is a basis for many speculations in astrophysics and cosmology.

But! Assume again that the neutrino source is $\pi_{\mu 2}$ decay and $M > m_\pi$. Then the transition $\nu_\alpha \to \nu_s$ in the pion rest frame is forbidden by the energy conservation.

$\Downarrow$

There must be some limitations & flaws in the QM formula. What are they?

Do CνB neutrinos oscillate?

The lightest (standard) relic neutrinos are most probably relativistic or perhaps even ultrarelativistic, while the heaviest ones can be subrelativistic. The QM approach is unable to work with such a set of ν states.

Does the motion of the neutrino source affect the transition probabilities?

To answer these and many similar questions

One has to unload the UR approximation & develop a covariant formalism.

In the QFT approach (on-shell regime): the effective (most probable) energies and momenta of virtual ν_is are *found to be* functions of the masses, most probable momenta and momentum spreads of all particles (wave packets) involved into the neutrino production/detection processes.

In particular, in the two limiting cases – ultrarelativistic (UR) and nonrelativistic (NR):

Ultrarelativistic case

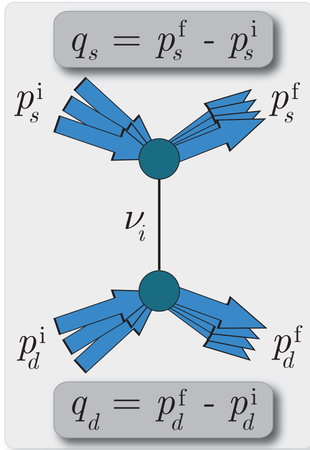
$(|q^0_{s,d}| \sim |\mathbf{q}_{s,d}| \gg m_i)$

$$\begin{cases} E_i = E_\nu \left[1 - \mathfrak{n} r_i - \mathfrak{m} r_i^2 + \ldots\right], \\ |\mathbf{p}_i| = E_\nu \left[1 - (\mathfrak{n}+1)\, r_i - \left(\mathfrak{m} + \mathfrak{n} + \dfrac{1}{2}\right) r_i^2 + \ldots\right], \\ v_i = 1 - r_i - \left(2\mathfrak{n} + \dfrac{1}{2}\right) r_i^2 + \ldots < 1, \end{cases}$$

Nonrelativistic case

$(|q^0_{s,d}| \sim m_i \gg |\mathbf{q}_{s,d}|)$

$$\begin{cases} E_i = m_i + \dfrac{m_i v_i^2}{2}\left(1 + \dfrac{3}{4}\delta_i + \ldots\right), \\ |\mathbf{p}_i| = m_i v_i \left(1 + \dfrac{1}{2}\delta_i + \ldots\right), \\ v_i \approx \dfrac{\boldsymbol{\varrho}_i \mathbf{l}}{1 + \varrho_i^0} \ll 1, \end{cases}$$



$$E_\nu \approx q_s^0 \approx -q_d^0, \quad r_i = \frac{m_i^2}{2E_\nu^2} \ll 1 \text{ (UR)},$$

$$\varrho_i^\mu = \frac{1}{m_i \mathscr{R}} \left[\widetilde{\Re}_s^{\mu 0}\left(m_i - q_s^0\right) + \widetilde{\Re}_d^{\mu 0}\left(m_i + q_d^0\right) - \widetilde{\Re}_s^{\mu k} q_s^k + \widetilde{\Re}_d^{\mu k} q_d^k\right], \quad |\varrho_i^\mu| \ll 1 \text{ (NR)}.$$

Definite momentum assumption

In the naive QM approach, the assumed definite momenta of neutrinos (both ν_α and ν_i) imply that the spatial coordinates of neutrino production ($\mathbf{X}_s$) and detection ($\mathbf{X}_d$) are fully uncertain (Heisenberg's principle).

$\Downarrow$

The distance $L = |\mathbf{X}_d - \mathbf{X}_s|$ is uncertain too, that makes the standard QM formula for the flavor transition probabilities to be formally speaking senseless.

In the correct theory, the neutrino momentum uncertainty $\delta|\mathbf{p}_\nu|$ must be at least of the order of $\min(1/D_s, 1/D_d)$, where D_s and D_d are the characteristic dimensions (along the neutrino beam) of the source and detector "machines".

$\Downarrow$

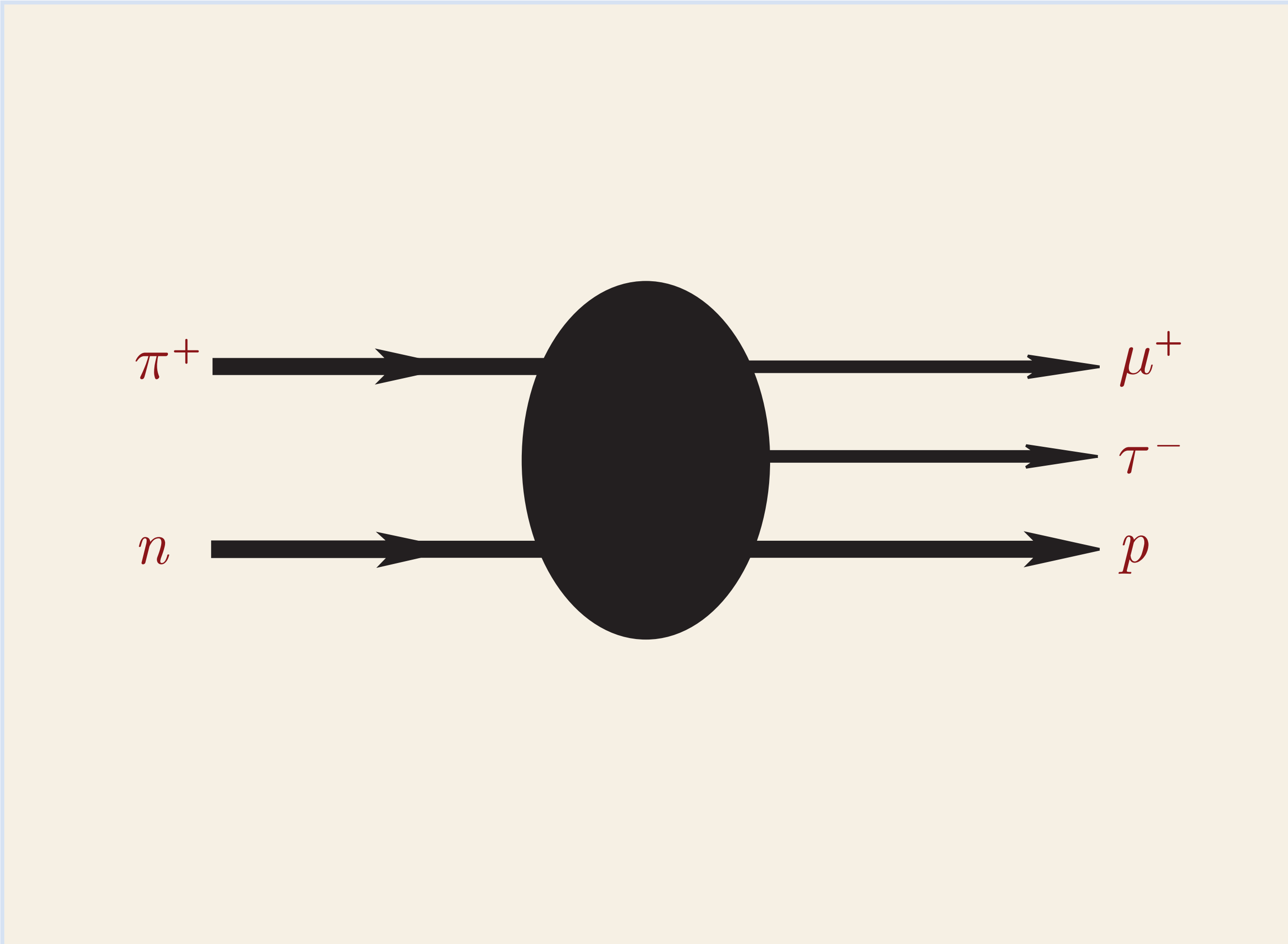
The neutrino states must be some wave packets (WP) [though having very small spreads] dependent, in general, on the quantum states of the particles [or, more exactly, also WPs] which participate in the production and detection processes.

In the on-shell QFT regime: the effective WPs of virtual UR ν_js are *found to be*

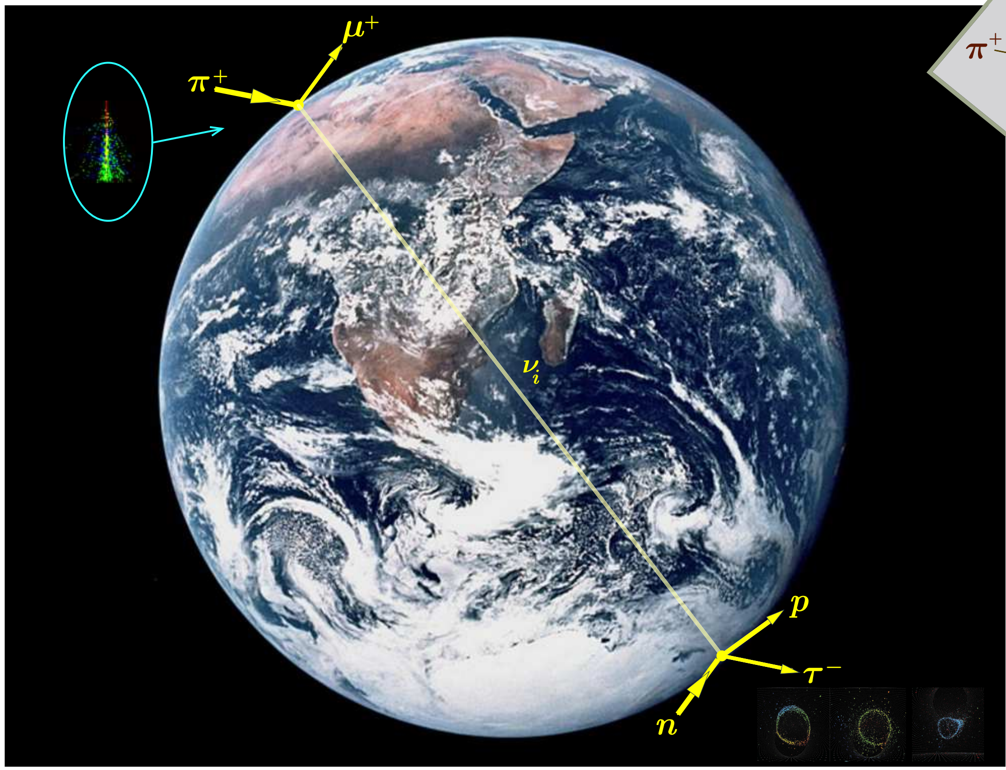
$$\psi_j^{(*)} = \exp\left\{\pm i(p_j X_{s,d}) - \frac{\widetilde{\mathfrak{D}}_j^2}{E_\nu^2}\left[(p_j X)^2 - m_j^2 X^2\right]\right\}, \qquad X = X_d - X_s,$$

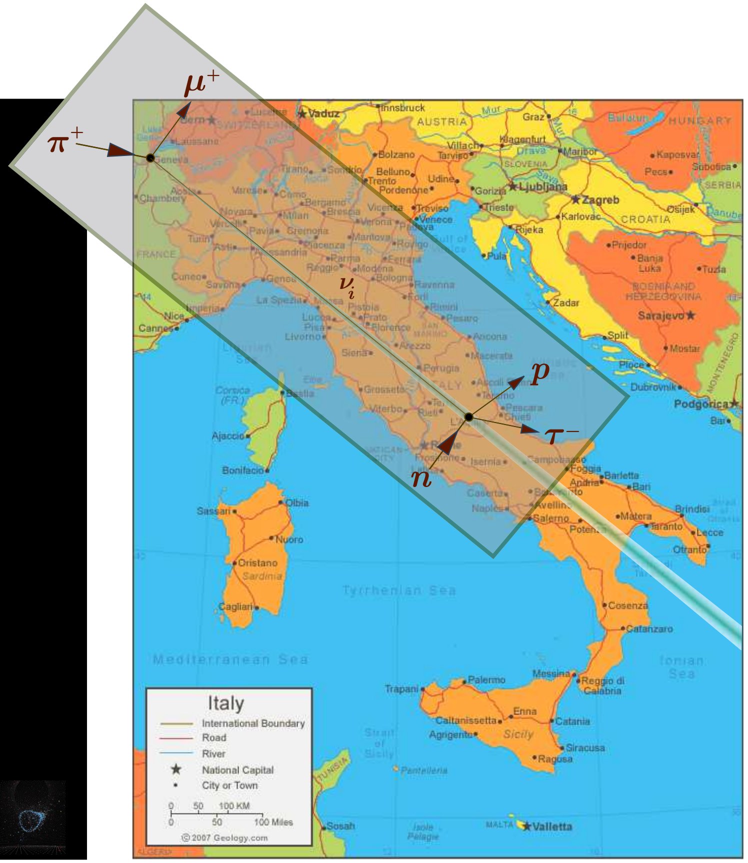
where $p_j = (E_j, \mathbf{p}_j)$ and $X_{s,d}$ are the 4-vectors which characterize the space-time location of the ν production and detection processes, while $\widetilde{\mathfrak{D}}_j$ are certain (in general, complex-valued) functions of the masses, mean momenta and momentum spreads of all particles involved into these processes. [$\widetilde{\mathfrak{D}}_j/E_\nu$ and thereby ψ_j are Lorentz invariants.]

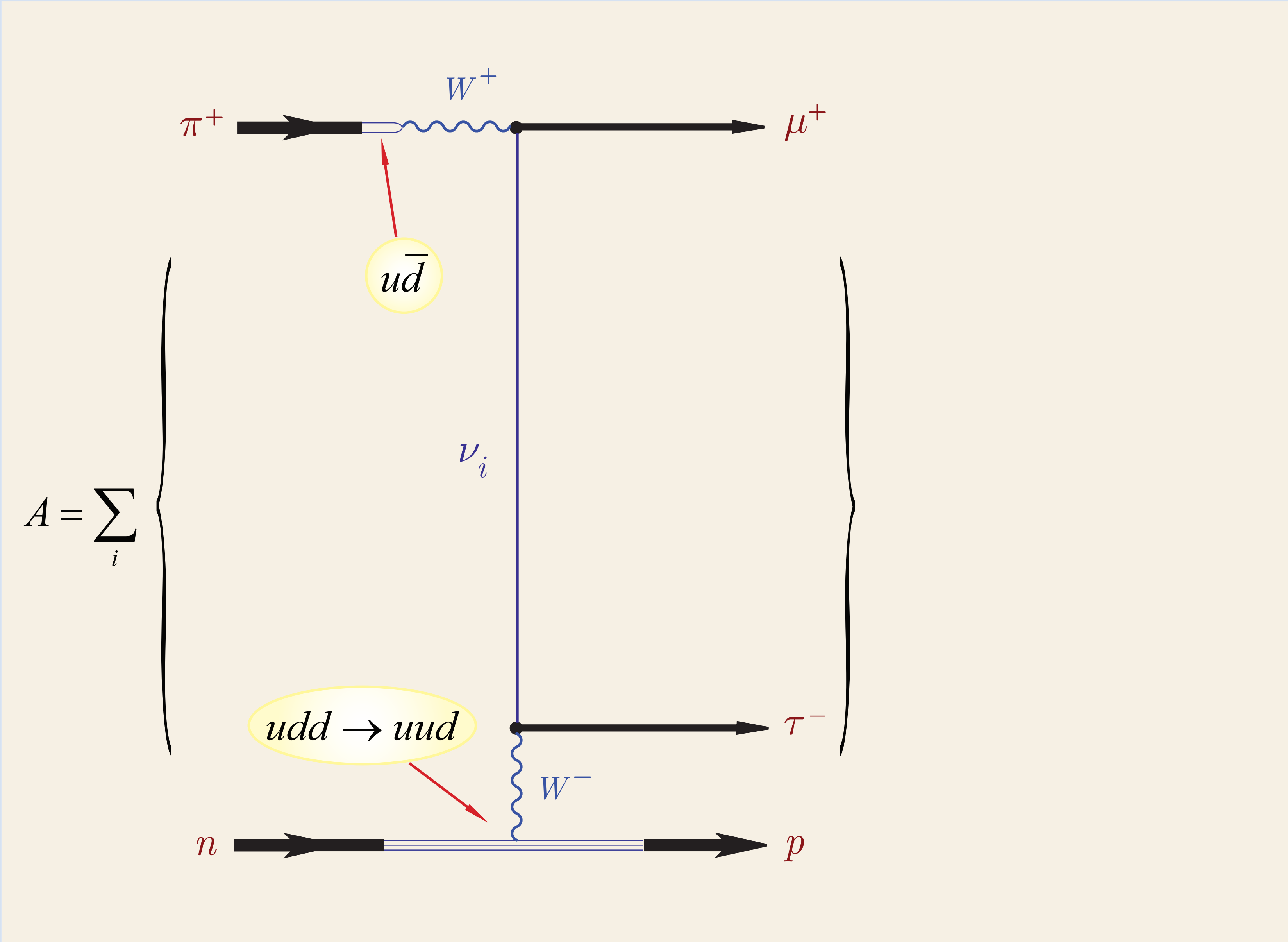
QFT approach by the example of the reaction $\pi \oplus n \to \mu \oplus \tau p$



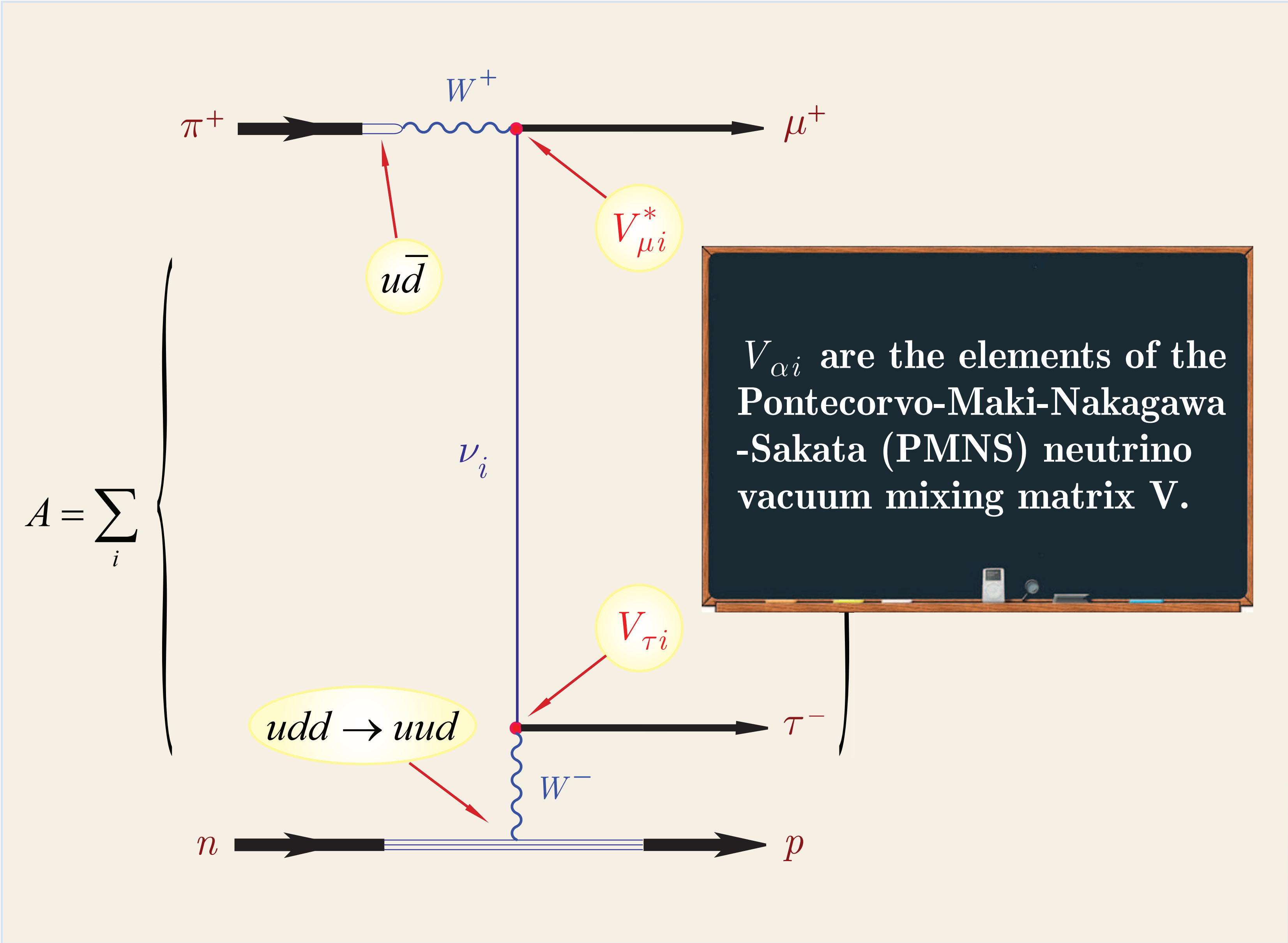
The rare reactions $\pi^+ \oplus n \to \mu^+ \oplus \tau^- p + \ldots$ were (indirectly) detected by several underground experiments (Kamiokande, IMB, Super-Kamiokande) with atmospheric neutrinos. In 2010, OPERA experiment (INFN, LNGS) with the CNGS neutrino beam announced the direct observation of the first τ^- candidate event.

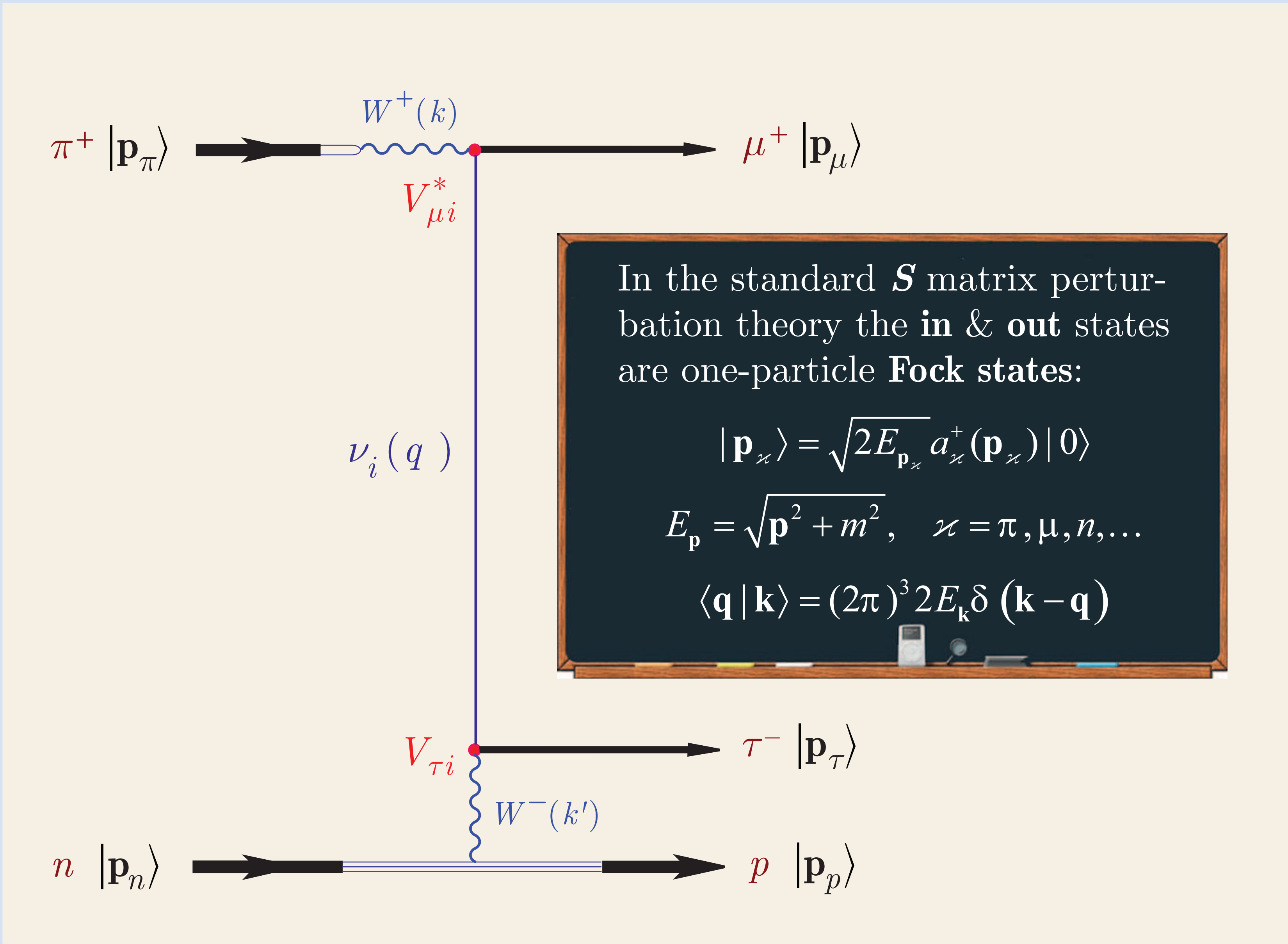


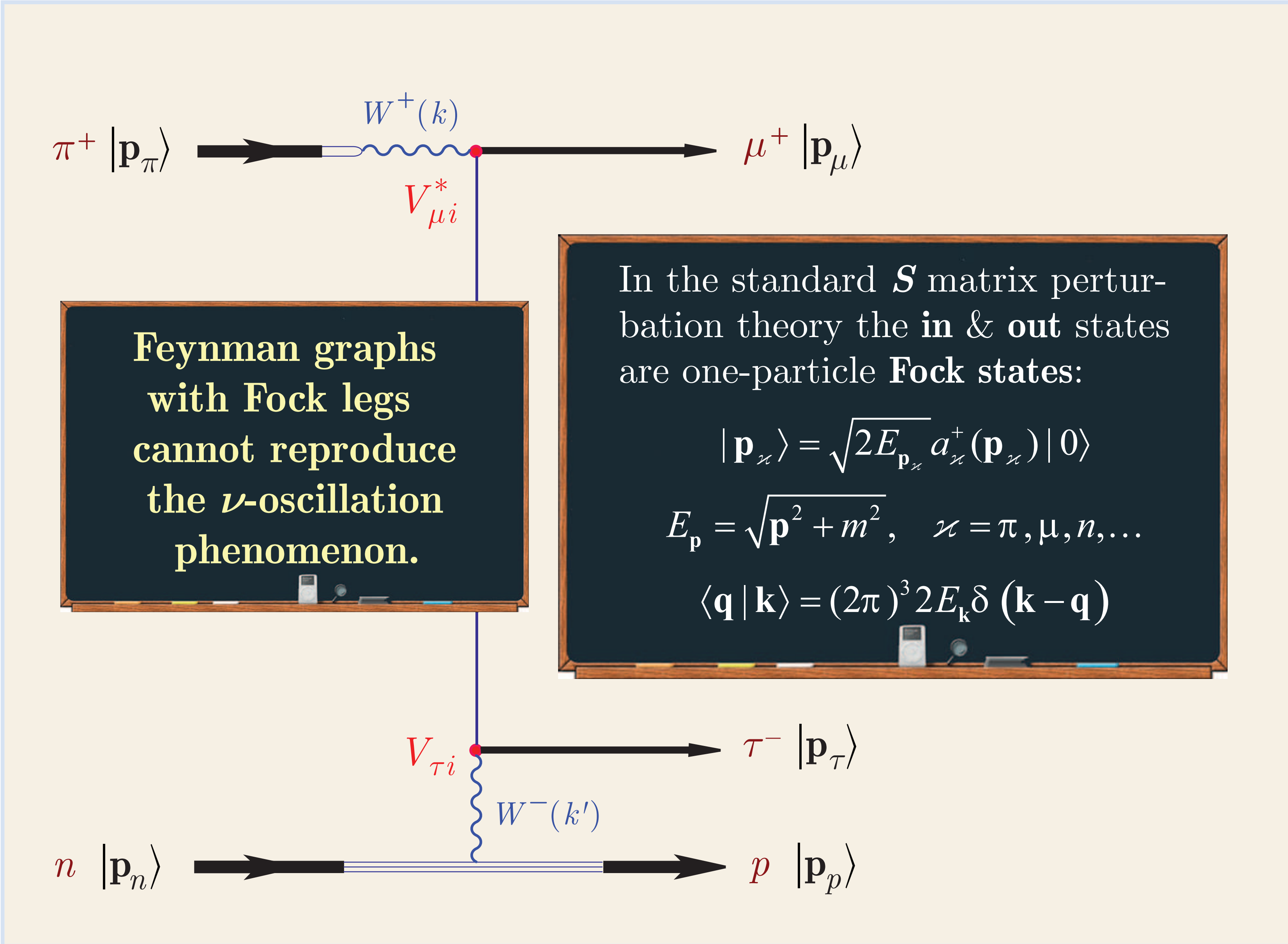




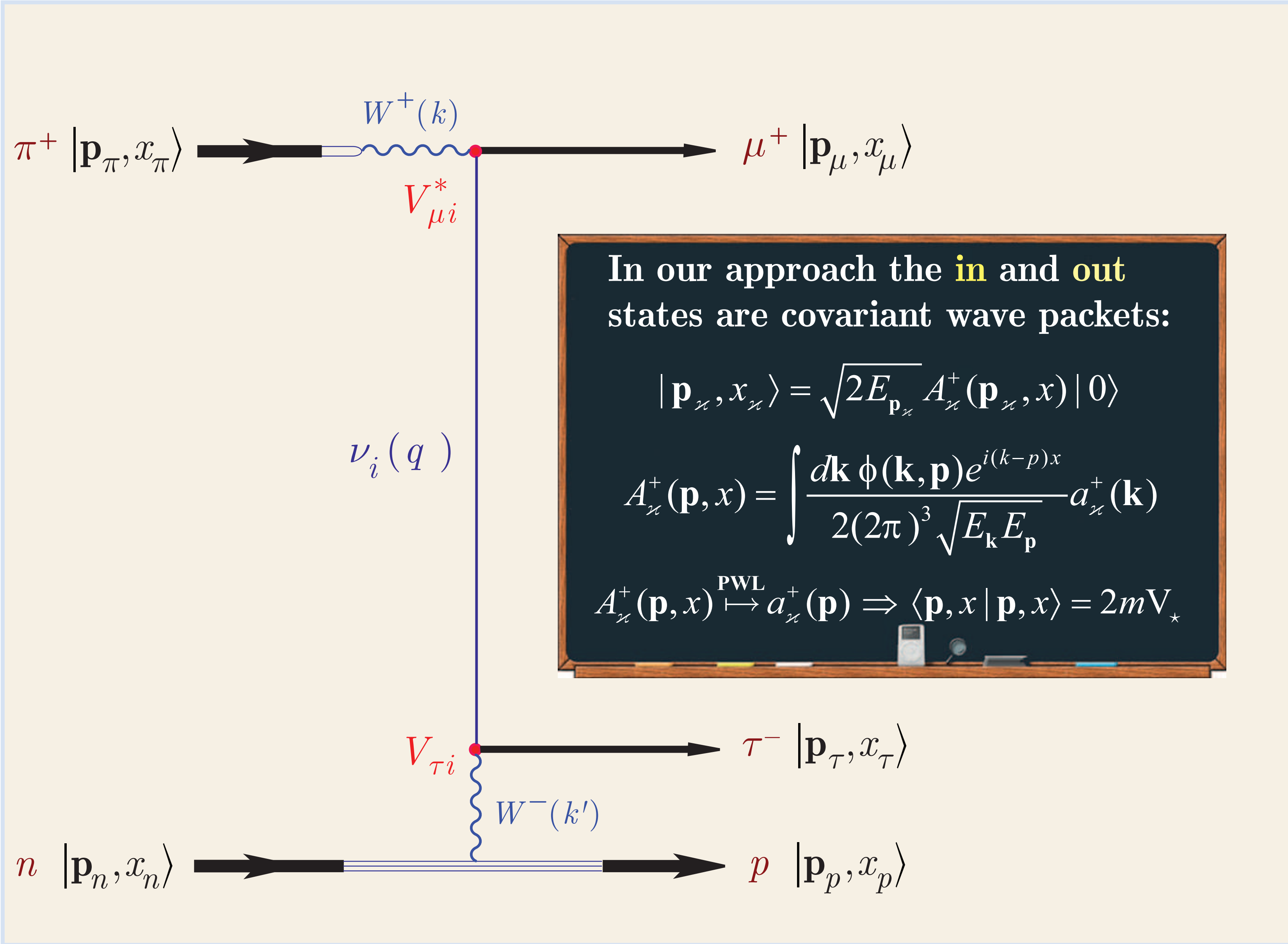
W^+
π^+
μ^+
$u\bar{d}$
$A = \sum_i$
ν_i
$udd \to uud$
τ^-
W^-
n
p

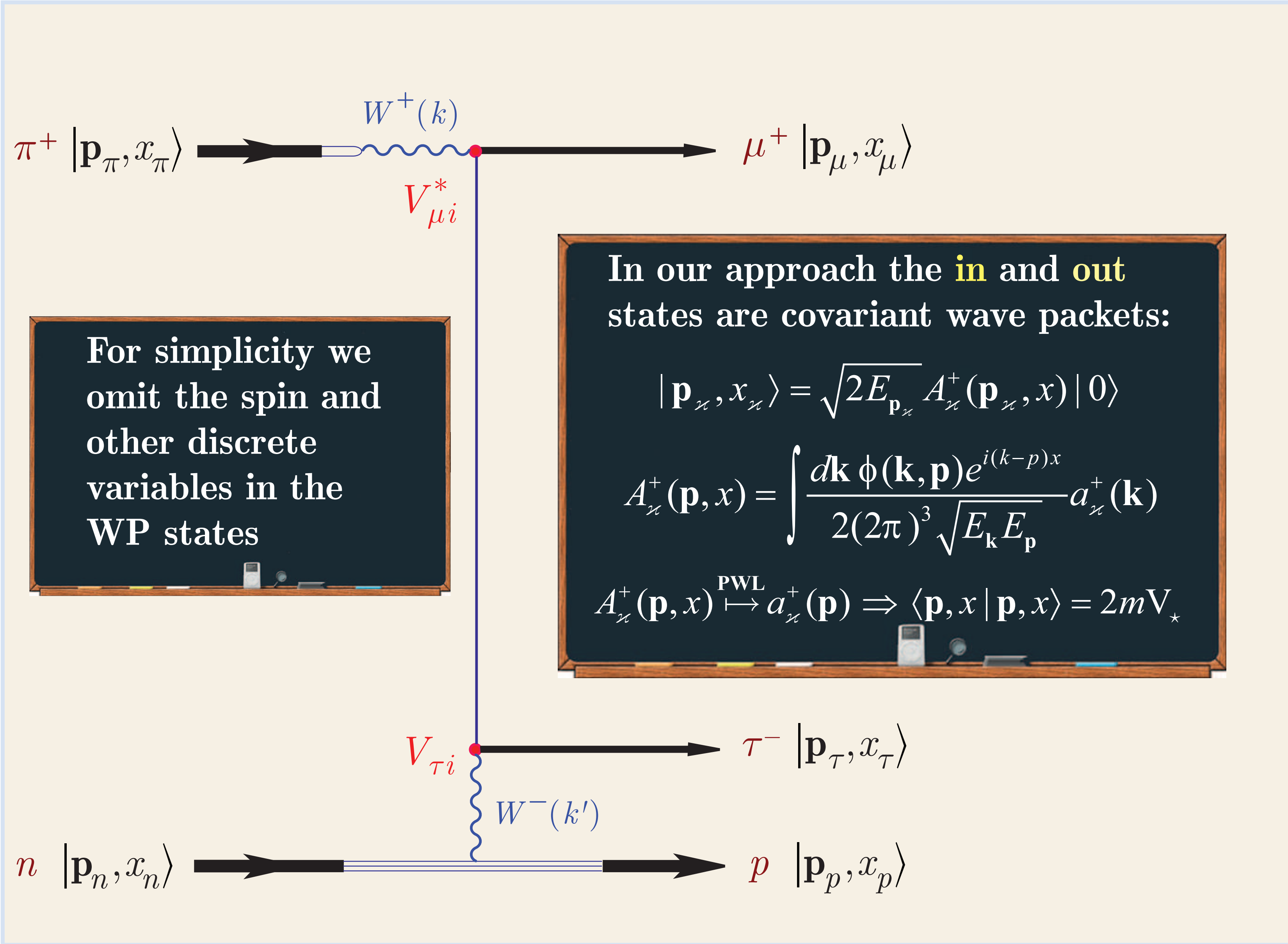

W^+
π^+
μ^+
$u\bar{d}$
$V^*_{\mu i}$
ν_i
$A = \sum_i$
$V_{\alpha i}$ are the elements of the Pontecorvo-Maki-Nakagawa -Sakata (PMNS) neutrino vacuum mixing matrix V.
$V_{\tau i}$
$udd \to uud$
τ^-
W^-
n
p


$W^+(k)$
$\pi^+ \ |\mathbf{p}_\pi\rangle$
$\mu^+ \ |\mathbf{p}_\mu\rangle$
$V^*_{\mu i}$
$\nu_i(q)$
In the standard S matrix perturbation theory the **in** & **out** states are one-particle **Fock states**:
$|\mathbf{p}_\varkappa\rangle = \sqrt{2E_{\mathbf{p}_\varkappa}}\, a^+_\varkappa(\mathbf{p}_\varkappa)\,|0\rangle$
$E_\mathbf{p} = \sqrt{\mathbf{p}^2 + m^2}, \quad \varkappa = \pi, \mu, n, \ldots$
$\langle \mathbf{q}\,|\,\mathbf{k}\rangle = (2\pi)^3 2E_\mathbf{k}\delta(\mathbf{k} - \mathbf{q})$
$V_{\tau i}$
$\tau^- \ |\mathbf{p}_\tau\rangle$
$W^-(k')$
$n \ |\mathbf{p}_n\rangle$
$p \ |\mathbf{p}_p\rangle$

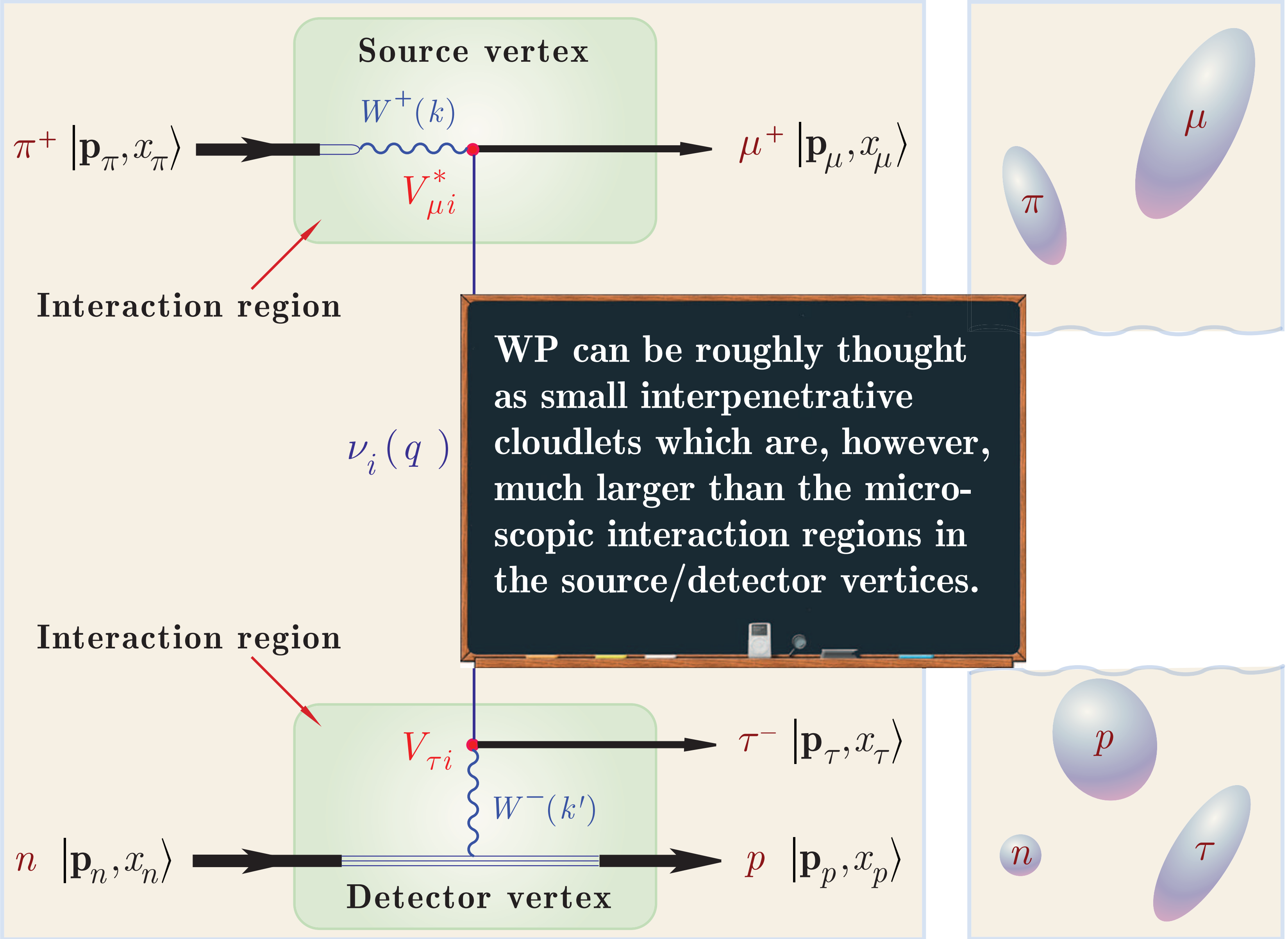


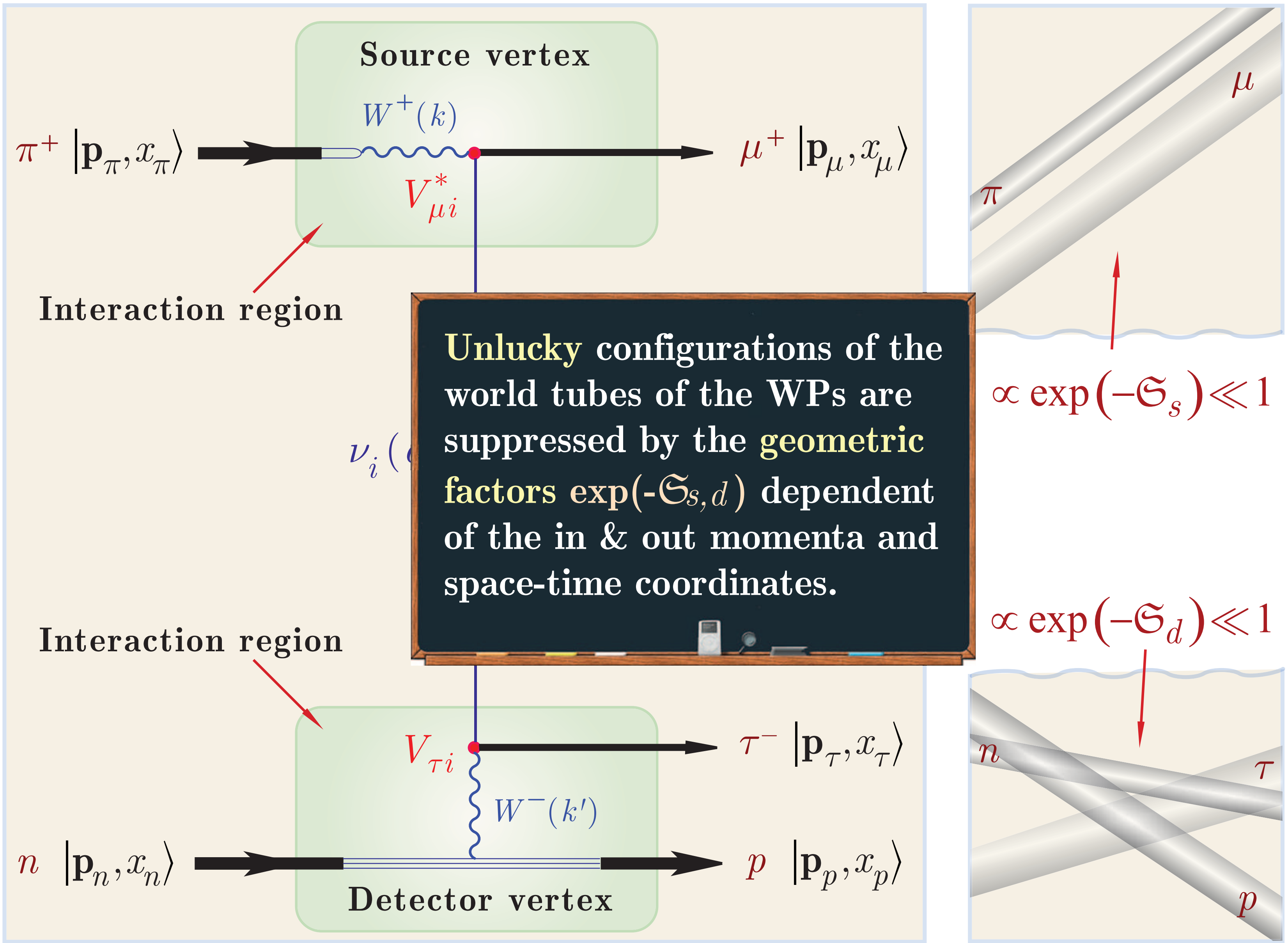
$W^+(k)$
$\pi^+ \; |\mathbf{p}_\pi\rangle$
$\mu^+ \; |\mathbf{p}_\mu\rangle$
$V^*_{\mu i}$
Feynman graphs with Fock legs cannot reproduce the ν-oscillation phenomenon.
In the standard S matrix perturbation theory the **in** & **out** states are one-particle **Fock states**:
$|\mathbf{p}_\varkappa\rangle = \sqrt{2E_{\mathbf{p}_\varkappa}}\, a^+_\varkappa(\mathbf{p}_\varkappa)\,|0\rangle$
$E_\mathbf{p} = \sqrt{\mathbf{p}^2 + m^2}, \quad \varkappa = \pi, \mu, n, \ldots$
$\langle \mathbf{q}\,|\,\mathbf{k}\rangle = (2\pi)^3\, 2E_\mathbf{k}\,\delta\left(\mathbf{k} - \mathbf{q}\right)$
$V_{\tau i}$
$\tau^- \; |\mathbf{p}_\tau\rangle$
$W^-(k')$
$n \; |\mathbf{p}_n\rangle$
$p \; |\mathbf{p}_p\rangle$

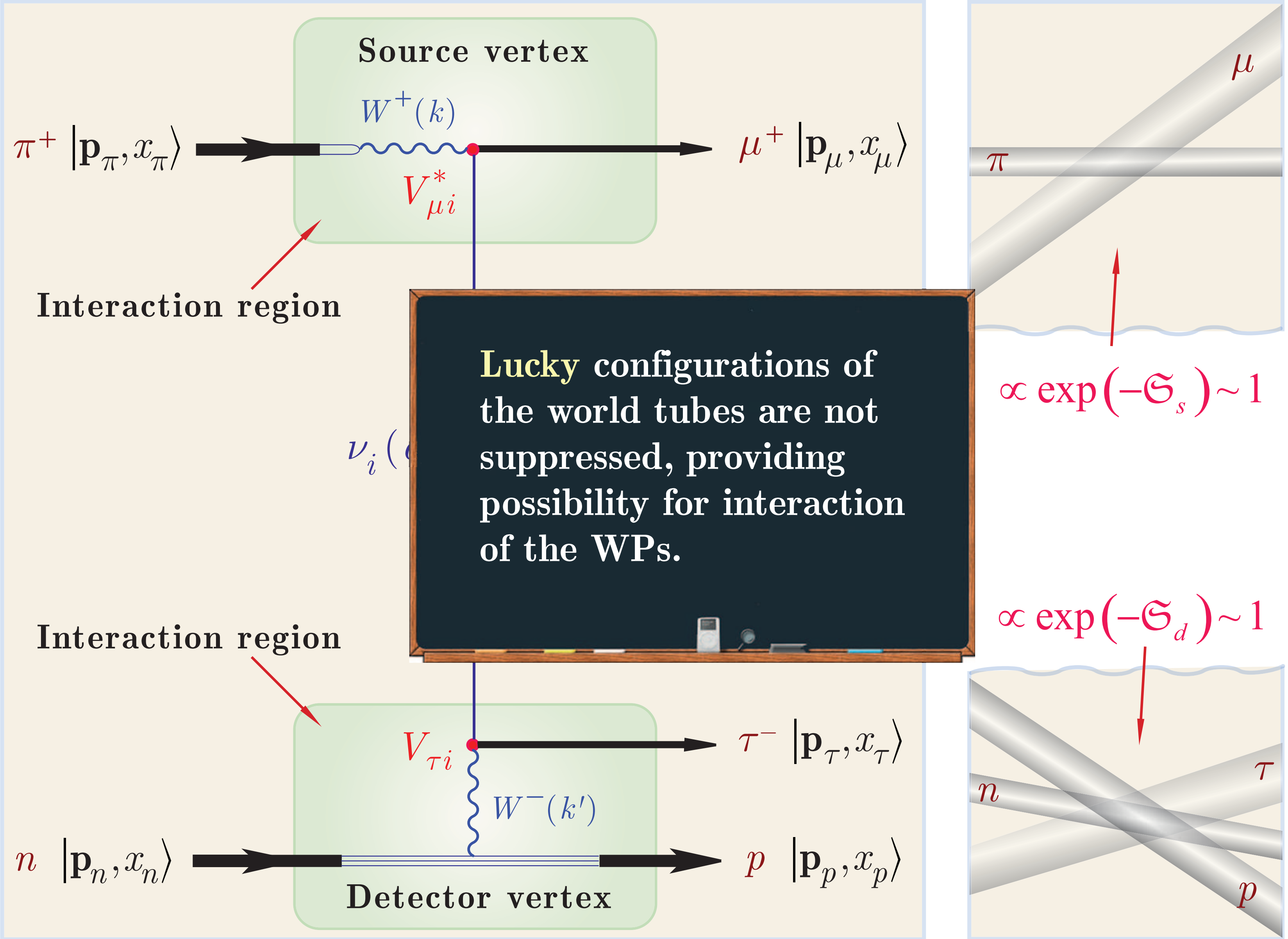

$\pi^+ \; |\mathbf{p}_\pi, x_\pi\rangle$
$W^+(k)$
$\mu^+ \; |\mathbf{p}_\mu, x_\mu\rangle$
$V^*_{\mu i}$
$\nu_i(q)$
$V_{\tau i}$
$\tau^- \; |\mathbf{p}_\tau, x_\tau\rangle$
$W^-(k')$
$n \; |\mathbf{p}_n, x_n\rangle$
$p \; |\mathbf{p}_p, x_p\rangle$
In our approach the in and out states are covariant wave packets:
$|\mathbf{p}_\varkappa, x_\varkappa\rangle = \sqrt{2E_{\mathbf{p}_\varkappa}}\, A^+_\varkappa(\mathbf{p}_\varkappa, x)\,|0\rangle$
$A^+_\varkappa(\mathbf{p}, x) = \int \frac{d\mathbf{k}\, \phi(\mathbf{k},\mathbf{p}) e^{i(k-p)x}}{2(2\pi)^3 \sqrt{E_\mathbf{k} E_\mathbf{p}}} a^+_\varkappa(\mathbf{k})$
$A^+_\varkappa(\mathbf{p}, x) \overset{\text{PWL}}{\longmapsto} a^+_\varkappa(\mathbf{p}) \Rightarrow \langle \mathbf{p}, x | \mathbf{p}, x\rangle = 2m\mathrm{V}_\star$

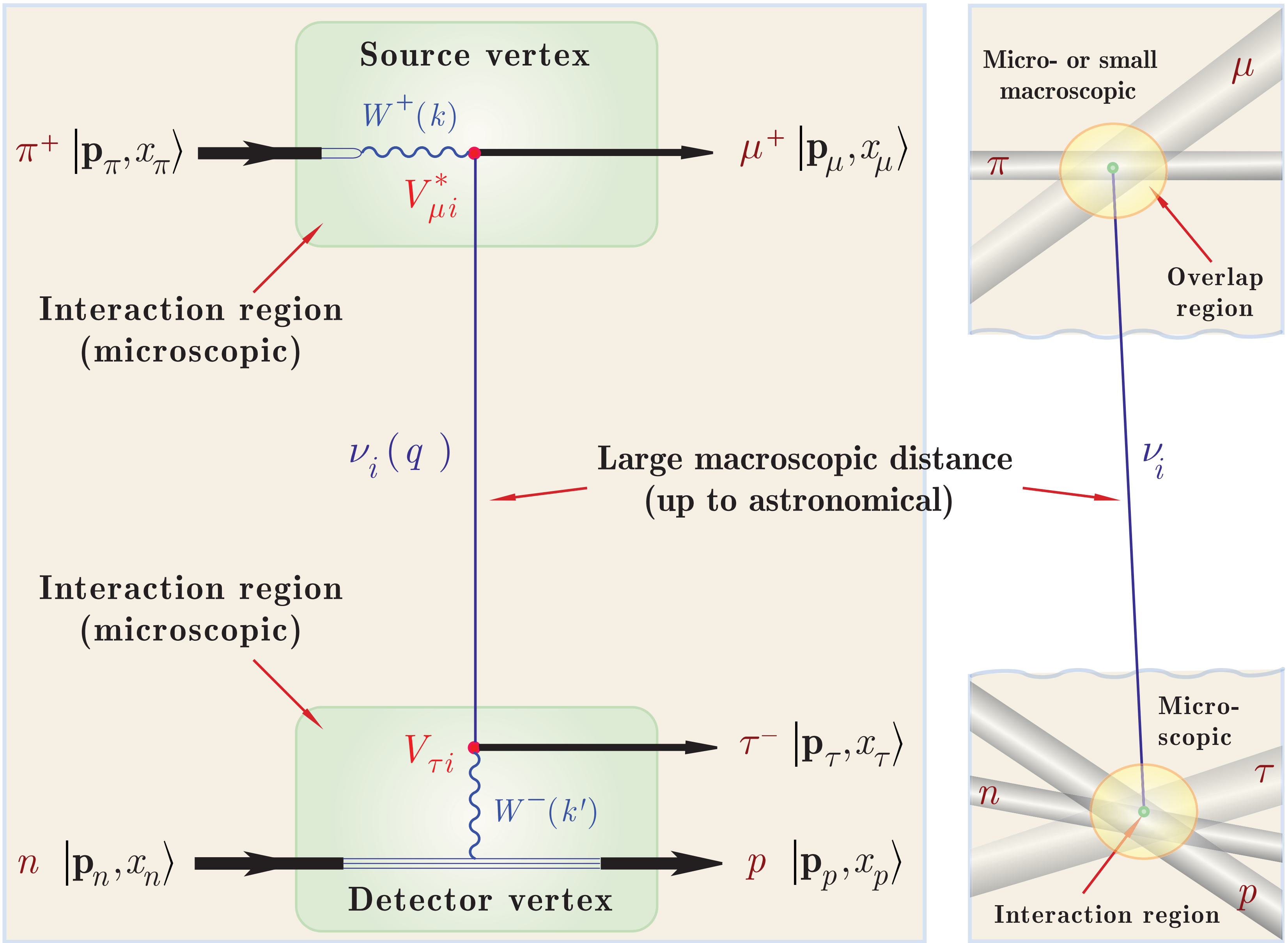


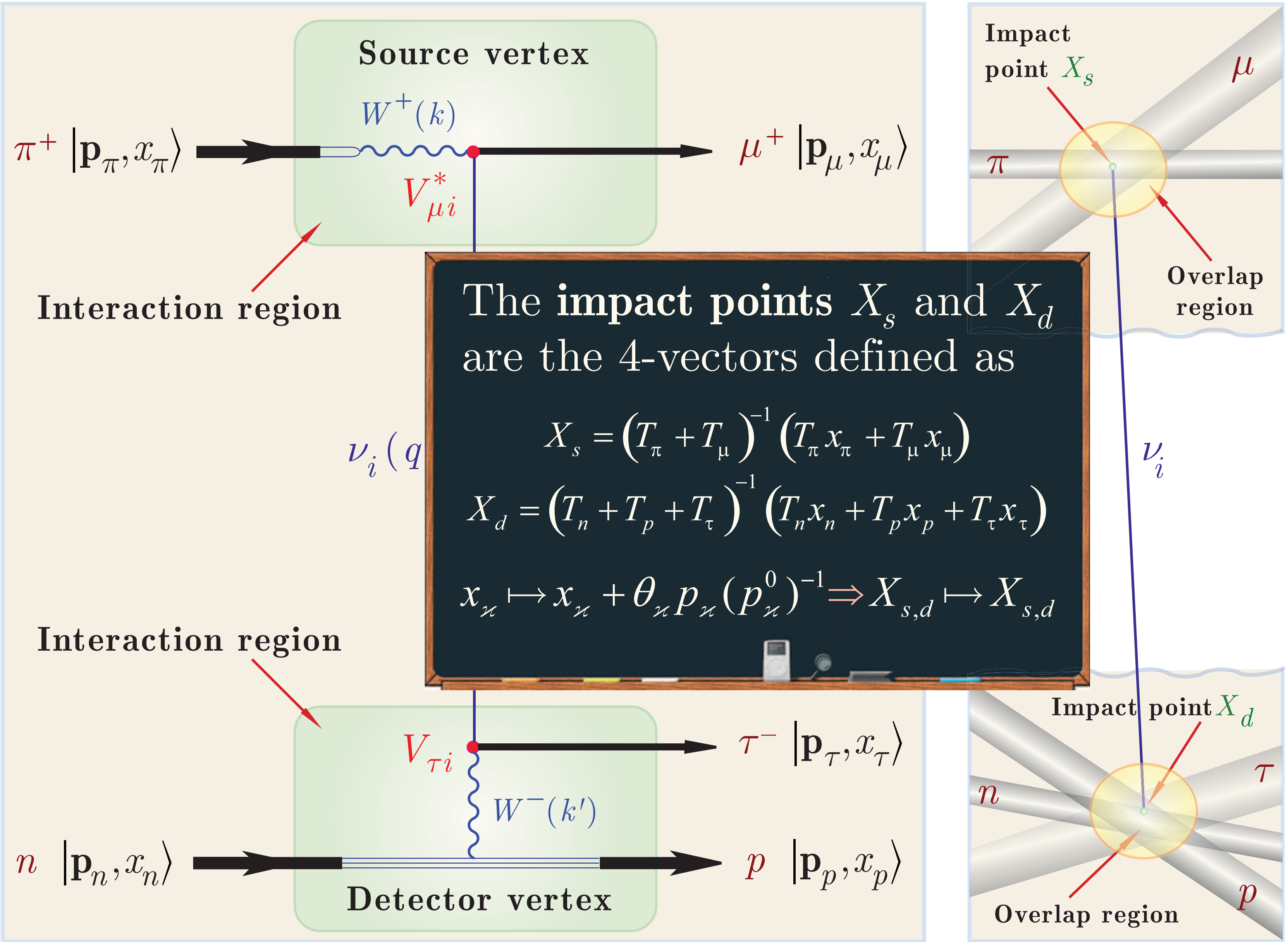
$\pi^+ \; |\mathbf{p}_\pi, x_\pi\rangle$
$W^+(k)$
$\mu^+ \; |\mathbf{p}_\mu, x_\mu\rangle$
$V^*_{\mu i}$
In our approach the in and out states are covariant wave packets:
$|\mathbf{p}_\varkappa, x_\varkappa\rangle = \sqrt{2E_{\mathbf{p}_\varkappa}}\, \mathbb{A}^+_\varkappa(\mathbf{p}_\varkappa, x)\,|0\rangle$
$\mathbb{A}^+_\varkappa(\mathbf{p}, x) = \int \frac{d\mathbf{k}\, \phi(\mathbf{k}, \mathbf{p}) e^{i(k-p)x}}{2(2\pi)^3 \sqrt{E_\mathbf{k} E_\mathbf{p}}} a^+_\varkappa(\mathbf{k})$
$\mathbb{A}^+_\varkappa(\mathbf{p}, x) \overset{\text{PWL}}{\longmapsto} a^+_\varkappa(\mathbf{p}) \Rightarrow \langle \mathbf{p}, x | \mathbf{p}, x\rangle = 2m\mathrm{V}_\star$
For simplicity we omit the spin and other discrete variables in the WP states
$V_{\tau i}$
$\tau^- \; |\mathbf{p}_\tau, x_\tau\rangle$
$W^-(k')$
$n \; |\mathbf{p}_n, x_n\rangle$
$p \; |\mathbf{p}_p, x_p\rangle$


Source vertex
$W^+(k)$
$\pi^+\ |\mathbf{p}_\pi, x_\pi\rangle$
$\mu^+\ |\mathbf{p}_\mu, x_\mu\rangle$
$V^*_{\mu i}$
Interaction region
$\nu_i(q)$
WP can be roughly thought as small interpenetrative cloudlets which are, however, much larger than the micro-scopic interaction regions in the source/detector vertices.
Interaction region
$V_{\tau i}$
$\tau^-\ |\mathbf{p}_\tau, x_\tau\rangle$
$W^-(k')$
$n\ |\mathbf{p}_n, x_n\rangle$
$p\ |\mathbf{p}_p, x_p\rangle$
Detector vertex
π
μ
p
n
τ


Source vertex
$W^+(k)$
$\pi^+ \, |\mathbf{p}_\pi, x_\pi\rangle$
$\mu^+ \, |\mathbf{p}_\mu, x_\mu\rangle$
$V^*_{\mu i}$
Interaction region
Unlucky configurations of the world tubes of the WPs are suppressed by the geometric factors exp(-$\mathfrak{S}_{s,d}$) dependent of the in & out momenta and space-time coordinates.
$\propto \exp(-\mathfrak{S}_s) \ll 1$
π
μ
$\propto \exp(-\mathfrak{S}_d) \ll 1$
Interaction region
$V_{\tau i}$
$\tau^- \, |\mathbf{p}_\tau, x_\tau\rangle$
$W^-(k')$
$n \, |\mathbf{p}_n, x_n\rangle$
$p \, |\mathbf{p}_p, x_p\rangle$
Detector vertex
n
τ
p


Source vertex
$W^+(k)$
$\pi^+ \; |\mathbf{p}_\pi, x_\pi\rangle$
$\mu^+ \; |\mathbf{p}_\mu, x_\mu\rangle$
$V^*_{\mu i}$
Interaction region
ν_i
Lucky configurations of the world tubes are not suppressed, providing possibility for interaction of the WPs.
Interaction region
$V_{\tau i}$
$\tau^- \; |\mathbf{p}_\tau, x_\tau\rangle$
$W^-(k')$
$n \; |\mathbf{p}_n, x_n\rangle$
$p \; |\mathbf{p}_p, x_p\rangle$
Detector vertex
μ
π
$\propto \exp(-\mathfrak{S}_s) \sim 1$
$\propto \exp(-\mathfrak{S}_d) \sim 1$
τ
n
p


Source vertex
$W^+(k)$
$\pi^+ \; |\mathbf{p}_\pi, x_\pi\rangle$
$\mu^+ \; |\mathbf{p}_\mu, x_\mu\rangle$
$V^*_{\mu i}$
Interaction region
(microscopic)
$\nu_i(q)$
Large macroscopic distance
(up to astronomical)
Interaction region
(microscopic)
$V_{\tau i}$
$\tau^- \; |\mathbf{p}_\tau, x_\tau\rangle$
$W^-(k')$
$n \; |\mathbf{p}_n, x_n\rangle$
$p \; |\mathbf{p}_p, x_p\rangle$
Detector vertex
Micro- or small
macroscopic
μ
π
Overlap
region
ν_i
Micro-
scopic
τ
n
p
Interaction region



Source vertex
$W^+(k)$
$\pi^+ \, |\mathbf{p}_\pi, x_\pi\rangle$
$\mu^+ \, |\mathbf{p}_\mu, x_\mu\rangle$
$V^*_{\mu i}$
Interaction region
$\nu_i\,(q$
The **impact points** X_s and X_d are the 4-vectors defined as
$X_s = (T_\pi + T_\mu)^{-1}(T_\pi x_\pi + T_\mu x_\mu)$
$X_d = (T_n + T_p + T_\tau)^{-1}(T_n x_n + T_p x_p + T_\tau x_\tau)$
$x_\varkappa \mapsto x_\varkappa + \theta_\varkappa p_\varkappa (p^0_\varkappa)^{-1} \Rightarrow X_{s,d} \mapsto X_{s,d}$
Interaction region
$V_{\tau i}$
$\tau^- \, |\mathbf{p}_\tau, x_\tau\rangle$
$W^-(k')$
$n \, |\mathbf{p}_n, x_n\rangle$
$p \, |\mathbf{p}_p, x_p\rangle$
Detector vertex
Impact point X_s
μ
π
Overlap region
ν_i
Impact point X_d
τ
n
p
Overlap region

Compendium

In the covariant WP approach there are several space-time scales:

- $\tau_I^{s,d}$ and $r_I^{s,d}$ – microscopic interaction time and radius defined by the Lagrangian.
- $\tau_O^{s,d}$ and $r_O^{s,d}$ – microscopic or small macroscopic dimensions of the overlap space-time regions of the interacting **in** and **out** packets in the source and detector vertices, defined by the effective dimensions of the packets.

  The suppression of the "unlucky" configurations of world tubes of the external packets is governed by the geometric factor in the amplitude:

  $$\exp\left[-\left(\mathfrak{S}_s + \mathfrak{S}_d\right)\right],$$

  where $\mathfrak{S}_{s,d}$ are the positive Lorentz and translation invariant functions of $\{\mathbf{p}_\varkappa\}$ and $\{x_\varkappa\}$. In the simplest one-parameter model of WP (relativistic Gaussian packet)

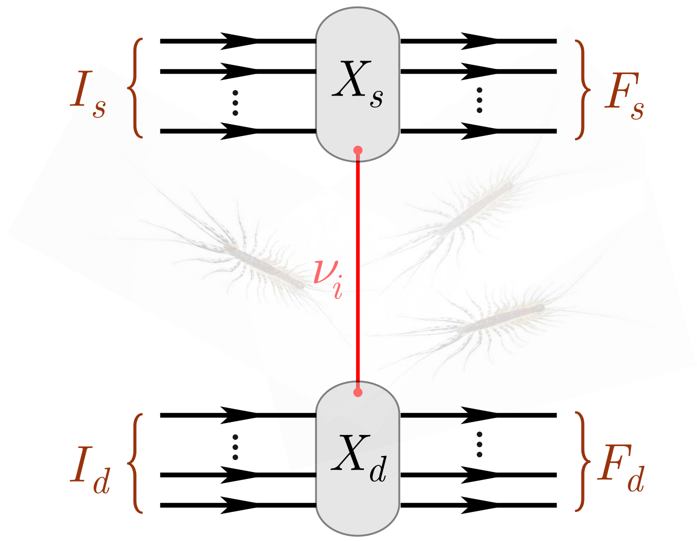
  $$\mathfrak{S}_{s,d} = \sum \sigma_\varkappa^2 \left|\boldsymbol{b}_\varkappa^\star\right|^2, \quad \varkappa \in S, D,$$

  where $\sigma_\varkappa$ are the momentum speeds of the packet $\varkappa$ and $\boldsymbol{b}_\varkappa^\star$ is the classical impact vector in the rest frame of the packet $\varkappa$ relative to the corresponding impact point.

- $T = X_d^0 - X_s^0$ and $L = |\mathbf{X}_d - \mathbf{X}_s|$ – large macroscopic neutrino time of flight and way between the impact points X_s and X_d.
  For light neutrinos, the impact points lie very close to the light cone $T^2 = L^2$.
- In usual circumstance (terrestrial experiments) $\tau_I^{s,d} \ll \tau_O^{s,d} \ll T$ and $r_I^{s,d} \ll r_O^{s,d} \ll L$.

Feynman rules for multipeds

We will deal with the generic connected diagrams. ▷



- The legs correspond to asymptotically free incoming ("in") and outgoing ("out") WPs in the coordinate representation. Here and below: I_s (F_s) is the set of in (out) packets in the block X_s ("source"), I_d (F_d) is the set of in (out) packets in the block X_d ("detector").
- The internal line denotes the causal Green's function of the neutrino mass eigenfield ν_i ($i = 1, 2, 3, \ldots$). The blocks X_s and X_d are assumed to be macroscopically separated.
- For narrow enough WPs, the Feynman rules for the legs are to be modified in a trivial way:

$$\langle 0|\Psi(y)|\mathbf{p}, s\rangle \;\longmapsto\; \langle 0|\Psi(y)|\mathbf{p}, s, x\rangle \approx e^{-ipx} u_s(\mathbf{p})\psi(\mathbf{p}, x-y), \tag{1}$$

where $\Psi(y)$ is the relevant free field operator [in Eq. (1), the spin-$\frac{1}{2}$ fermion field is used as an example] and $\psi(\mathbf{p}, x)$ is the Lorentz-invariant function,

$$\psi(\mathbf{p}, x) = \int \frac{d\mathbf{k}}{(2\pi)^3 2E_{\mathbf{k}}} e^{ikx} \phi(\mathbf{k}, \mathbf{p}) = \psi(\mathbf{0}, x_\star),$$

satisfying the Klein-Gordon equation, $(\Box_x - m^2)\psi(\mathbf{p}, x) = 0$. [Therefore it is a relativistic wave packet in terms of conventional (axiomatic) scattering theory.]

For spinor field, the approximation (1) is valid under the following condition

$$|i\nabla_{\mathbf{y}} \ln \psi(\mathbf{p}, x-y) + \mathbf{p}| \ll 2E_{\mathbf{p}}.$$

It is, in fact, one the basic (most limiting) approximations in the whole formalism. The approximate relations analogous to (1) take place for the free fields of any spin, providing us with the modified Feynman-rule factors for the external lines of any diagram. In particular, the relation (1) is exact for the scalar and pseudoscalar fields $\Phi(x)$:

$$\langle 0|\Phi(y)|\mathbf{p}, s=0, x\rangle = e^{-ipx}\psi(\mathbf{p}, x-y)$$

- As a result for each external line, the *standard* (plain-wave) Feynman factor must be multiplied by

$$\begin{array}{ll} e^{-ip_a(x_a-y)}\psi_a\left(\mathbf{p}_a, x_a-y\right) & \text{for } a \in I_s \oplus I_d \\ \text{or} & \\ e^{+ip_b(x_b-y)}\psi_b^*\left(\mathbf{p}_b, x_b-y\right) & \text{for } b \in F_s \oplus F_d, \end{array}$$

where each function $\psi_\varkappa\left(\mathbf{p}_\varkappa, y\right)$ ($\varkappa = a, b$) is specified by the mass $m_\varkappa$ and the set of momentum spreads $\boldsymbol{\sigma}_\varkappa = \{\sigma_{1\varkappa}, \sigma_{2\varkappa}, \ldots\}$.

Generally the set $\boldsymbol{\sigma}_\varkappa$ forms a tensor with respect to Lorentz transformations. But to simplify matters, below we'll only discuss a model with one scalar spread parameter $\sigma_\varkappa$.

- The internal lines and loops in the diagram remain unchanged.

Relativistic Gaussian packets (RGP)

In further consideration we will use a simple model of the QFT WP state – relativistic Gaussian packet (RGP), in which the form-factor function $\phi(\mathbf{k},\mathbf{p})$ is of the form

$$\phi(\mathbf{k},\mathbf{p}) = \frac{2\pi^2}{\sigma^2 K_1(m^2/2\sigma^2)} \exp\left(-\frac{E_{\mathbf{k}}E_{\mathbf{p}} - \mathbf{kp}}{2\sigma^2}\right) \stackrel{\text{def}}{=} \phi_G(\mathbf{k},\mathbf{p}), \tag{2}$$

where K_1 is the modified Bessel function of the 3rd kind of order 1.

$$K_1(z) = z\int_1^\infty dte^{-zt}\sqrt{t^2-1} \qquad \left(|\arg z| < \frac{\pi}{2}\right).$$

One may easily check that the function (2) has the correct plane-wave limit and satisfies the normalization conditions. In what follows we assume $\sigma^2 \ll m^2$ [invariant condition of the tightness]. Then the function (2) can be rewritten as the asymptotic expansion in σ^2/m^2:

$$\phi_G(\mathbf{k},\mathbf{p}) = \frac{2\pi^{3/2}}{\sigma^2}\frac{m}{\sigma}\exp\left[\frac{(k-p)^2}{4\sigma^2}\right]\left[1+\frac{3\sigma^2}{4m^2}+\mathcal{O}\left(\frac{\sigma^4}{m^4}\right)\right].$$

In the nonrelativistic case, $(|\mathbf{k}|+|\mathbf{p}|)^2 \ll 4m^2$, and only in this case this form factor coincides, up to a normalization factor, with the widely used (noncovariant) Gaussian distribution:

$$\varphi_G(\mathbf{k}-\mathbf{p}) \propto \exp\left[-\frac{(\mathbf{k}-\mathbf{p})^2}{4\sigma^2}\right].$$

Exact wavefunction $\psi(\mathbf{p}, x)$ for RGP

$$\psi(\mathbf{p}, x) = \frac{K_1(\zeta m^2/2\sigma^2)}{\zeta K_1(m^2/2\sigma^2)} \stackrel{\text{def}}{=} \psi_G(\mathbf{p}, x),$$
$$\zeta = \sqrt{1 - \frac{4\sigma^2}{m^2}\left[\sigma^2 x^2 + i(px)\right]}.$$

Nondiffluent regime, contracted RGP

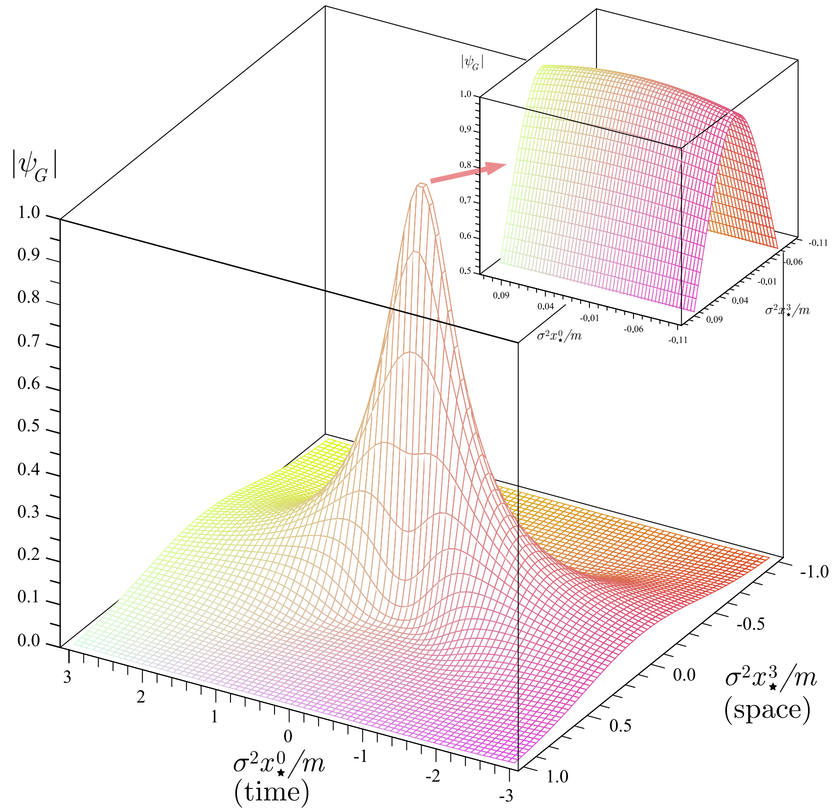
Under the following N&S conditions

$$\sigma^2 (x_\star^0)^2 \ll m^2/\sigma^2, \quad \sigma^2 |\mathbf{x}_\star|^2 \ll m^2/\sigma^2,$$
$$(px)^2 \ll m^4/\sigma^4, \quad (px)^2 - m^2 x^2 \ll m^4/\sigma^4$$

[two pairs of the inequalities are equivalent]
RGP is stable in its rest frame ($\mathbf{p}^\star = 0$):

$$\psi_G(\mathbf{0}, x^\star) = \exp\left(imx_\star^0 - \sigma^2 \mathbf{x}_\star^2\right).$$

In the lab. frame it has the following form:



3D plot of $|\psi_G(\mathbf{0}, x_\star)|$ vs. $\sigma^2 x_\star^0/m$ and $\sigma^2 x_\star^3/m$, assuming $\mathbf{x}_\star = (0, 0, x_\star^3)$ and $\sigma = 0.1m$.

$$\psi_G(\mathbf{p}, x) = \exp\left\{i(px) - (\sigma/m)^2\left[(px)^2 - m^2 x^2\right]\right\}.$$

Table: Maximum permissible values of σ ($\sigma_{\max} = \sqrt{m\Gamma}$, $\sigma \ll \sigma_{\max}$), the ratios $\Gamma/\sigma_{\max} = \sqrt{\Gamma/m}$, and the minimum permissible effective dimensions $d_\star^{\min} \approx 1.55/\sqrt{m\Gamma}$ in the CRGP approximation (contracted RGP) for the particles most relevant to neutrino production.

Particle	$\sigma_{\max}$ (eV)	$\Gamma/\sigma_{\max}$	$d_\star^{\min}$ (cm)
$\mu^\pm$	1.78×10^{-1}	1.68×10^{-9}	1.72×10^{-4}
$\tau^\pm$	2.01×10^{3}	1.13×10^{-6}	1.53×10^{-8}
$\pi^\pm$	1.88	1.35×10^{-8}	1.63×10^{-5}
π^0	3.25×10^{4}	2.41×10^{-4}	0.94×10^{-9}
$K^\pm$	5.12	1.04×10^{-8}	5.99×10^{-6}
K_S^0	6.05×10^{1}	1.22×10^{-7}	5.07×10^{-7}
K_L^0	2.53	5.08×10^{-9}	1.21×10^{-5}
$D^\pm$	1.09×10^{3}	5.82×10^{-7}	2.82×10^{-8}
D^0	1.73×10^{3}	9.28×10^{-7}	1.77×10^{-8}
$D_s^\pm$	1.61×10^{3}	8.18×10^{-7}	1.91×10^{-8}
$B^\pm$	1.46×10^{3}	2.76×10^{-7}	2.11×10^{-8}
B^0	1.51×10^{3}	2.86×10^{-7}	2.03×10^{-8}
B_s^0	1.55×10^{3}	2.89×10^{-7}	1.98×10^{-8}
n	2.64×10^{-5}	2.81×10^{-14}	1.16
Λ	5.28×10^{1}	4.74×10^{-7}	5.81×10^{-7}
$\Lambda_c^\pm$	2.74×10^{3}	1.87×10^{-6}	1.12×10^{-8}

The maximum permissible deviation of the mean mass of CRGP from the field mass, $\delta m = \overline{m} - m$, is equal to

$$\delta m_{\max} \approx \frac{3\sigma_{\max}^2}{2m} = 1.5\Gamma,$$

So, the correction to the field mass of the short-lived resonances can be essential, but for the long-lived particles we can (and we must) to neglect the weighting effect.

Calculation of a macroscopic amplitude

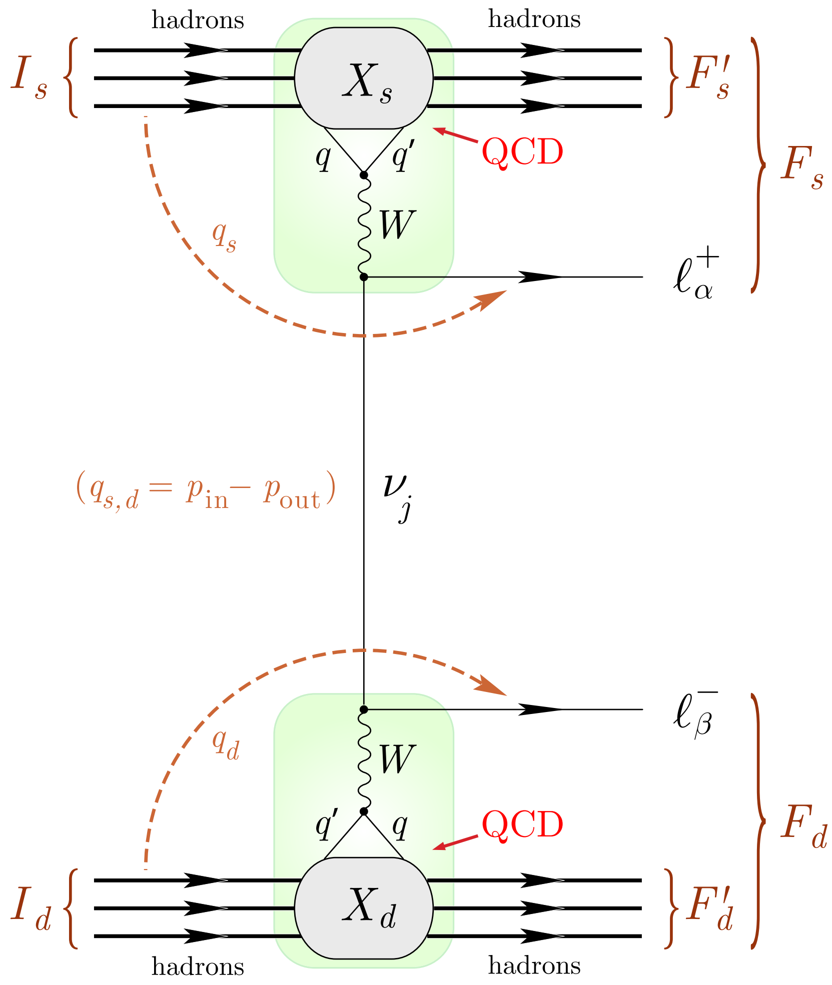
As a practically important (and very general) example, we consider the charged-current induced production of charged leptons ℓ_α^+ and ℓ_β^- (e, μ, τ) in the process

$$I_s \oplus I_d \to F_s' + \ell_\alpha^+ \oplus F_d' + \ell_\beta^-, \quad (3)$$

We assume for definiteness that all the external substates I_s, I_d, F_s', and F_d' consist exclusively of (asymptotically free) hadronic WPs. Consequently, if $\alpha \neq \beta$, the process (3) violates the lepton numbers L_α and L_β that is only possible via exchange of massive neutrinos (no matter whether they are Dirac or Majorana particles).

In the lowest nonvanishing (4-th) order in electroweak coupling, the process (3) is described by the sum of the diagrams shown in the figure.

The impact points X_s and X_d are macroscopically separated and all asymptotic conditions are assumed to be fulfilled.



A macroscopic Feynman diagram describing the flavor-violating process (3) with ν_j exchange.

The shortest possible sketch of the calculation

1. Quark-lepton blocks. We use the Standard Model (SM) phenomenologically extended by inclusion of a neutrino mass term. The quark-lepton blocks are described by the Lagrangian

$$\mathcal{L}_W(x) = -\frac{g}{2\sqrt{2}}\left[j_\ell(x)W(x) + j_q(x)W(x) + \text{H.c.}\right],$$

where g is the $SU(2)$ coupling constant, j_ℓ and j_q are the weak charged currents:

$$j_\ell^\mu(x) = \sum_{\alpha i} V_{\alpha i}^* \,\overline{\nu}_i(x) O^\mu \ell_\alpha(x), \qquad j_q^\mu(x) = \sum_{qq'} V_{qq'}^{\prime *}\,\overline{q}(x) O^\mu q'(x), \qquad [O^\mu = \gamma^\mu(1-\gamma_5)]\,.$$

Here $V_{\alpha i}$ ($\alpha = e, \mu, \tau$; $i = 1, 2, 3$) and $V'_{qq'}$ ($q = u, c, t$; $q' = d, s, b$) are the elements of the neutrino and quark mixing matrices ($\boldsymbol{V}$ and $\boldsymbol{V}'$, respectively).

The normalized amplitude is given by the 4th order of the perturbation theory in g:

$$\begin{aligned}\mathcal{A}_{\beta\alpha} &= \langle\mathbf{out}|\mathbb{S}|\mathbf{in}\rangle\left(\langle\mathbf{in}|\mathbf{in}\rangle\langle\mathbf{out}|\mathbf{out}\rangle\right)^{-1/2} \\ &= \frac{1}{\mathcal{N}}\left(\frac{-ig}{2\sqrt{2}}\right)^4 \langle F_s \oplus F_d|\, T\int dx dx' dy dy' : j_\ell(x)W(x) :: j_q(x')W^\dagger(x') : \\ &\quad\times : j_\ell^\dagger(y)W^\dagger(y) :: j_q^\dagger(y')W(y') : \mathbb{S}_h|I_s \oplus I_d\rangle. \end{aligned} \tag{4}$$

The normalization factor $\mathcal{N}$ in the CRGP approximation is given by

$$\mathcal{N}^2 = \langle\mathbf{in}|\mathbf{in}\rangle\langle\mathbf{out}|\mathbf{out}\rangle = \prod_{\varkappa \in I_s \oplus I_d \oplus F_s \oplus F_d} 2E_\varkappa \mathrm{V}_\varkappa(\boldsymbol{p}_\varkappa).$$

2. Hadronic blocks. The strong and (possibly) electromagnetic interactions responsible for nonperturbative processes of fragmentation and hadronization are described by the hadronic (QCD) interaction Lagrangian $\mathcal{L}_{\text{h}}(x)$ and the corresponding part of the full S-matrix is

$$\mathbb{S}_h = \exp\left[i\int dz\mathcal{L}_{\text{h}}(z)\right].$$

The following factorization theorem can be proved

$$\begin{aligned}\langle F_s'\oplus F_d'|T\left[: j_q^{\mu}(x) : \mathbb{S}_h : j_q^{\dagger\nu}(y) :\right]|I_s\oplus I_d\rangle &= \mathcal{J}_s^{\mu}(p_S)\mathcal{J}_d^{\nu\dagger}(p_D) \\ \times\Big[\prod_{a\in I_s} e^{-ip_ax_a}\psi_a(\boldsymbol{p}_a, x_a - x)\Big]&\Big[\prod_{b\in F_s'} e^{ip_bx_b}\psi_b^*(\boldsymbol{p}_b, x_b - x)\Big] \\ \times\Big[\prod_{a\in I_d} e^{-ip_ax_a}\psi_a(\boldsymbol{p}_a, x_a - y)\Big]&\Big[\prod_{b\in F_d'} e^{ip_bx_b}\psi_b^*(\boldsymbol{p}_b, x_b - y)\Big].\end{aligned}$$

Here $\mathcal{J}_s(p_S)$ and $\mathcal{J}_d(p_D)$ are the c-number hadronic currents in which the strong interactions are taken into account nonperturbatively, and p_S and p_D denote the sets of the momentum and spin variables of the hadronic states.

> The proof is based on the assumed narrowness of the WPs in the momentum space, macroscopic remoteness of the interaction regions in the source and detector vertices, and the consideration of translation invariance.

The explicit form of the hadronic currents $\mathcal{J}_s$ and $\mathcal{J}_d$ is not needed for our purposes.

By applying the new Feynman rules, factorization theorem, and other (both the standard QFT and speculative) tricks the amplitude (4) can be rewritten in the following way:

$$\mathcal{A}_{\beta\alpha} = \frac{g^4}{64\mathcal{N}} \sum_j V_{\beta j} \mathcal{J}_d^{\nu\dagger} \overline{u}(\boldsymbol{p}_\beta) O_{\nu'} \mathbb{G}_{\nu\mu}^{j\nu'\mu'} \left(\{\boldsymbol{p}_\varkappa, x_\varkappa\}\right) O_{\mu'} v(\boldsymbol{p}_\alpha) \mathcal{J}_s^\mu V_{\alpha j}^*, \tag{5}$$

$$\mathbb{G}_{\nu\mu}^{j\nu'\mu'} \left(\{\boldsymbol{p}_\varkappa, x_\varkappa\}\right) = \int \frac{dq}{(2\pi)^4} \mathbb{V}_d(q) \Delta_\nu^{\nu'}(q - p_\beta) \Delta^j(q) \Delta_\mu^{\mu'}(q + p_\alpha) \mathbb{V}_s(q). \tag{6}$$

Here $\mathbb{V}_s(q)$ and $\mathbb{V}_d(q)$ are the overlap integrals,

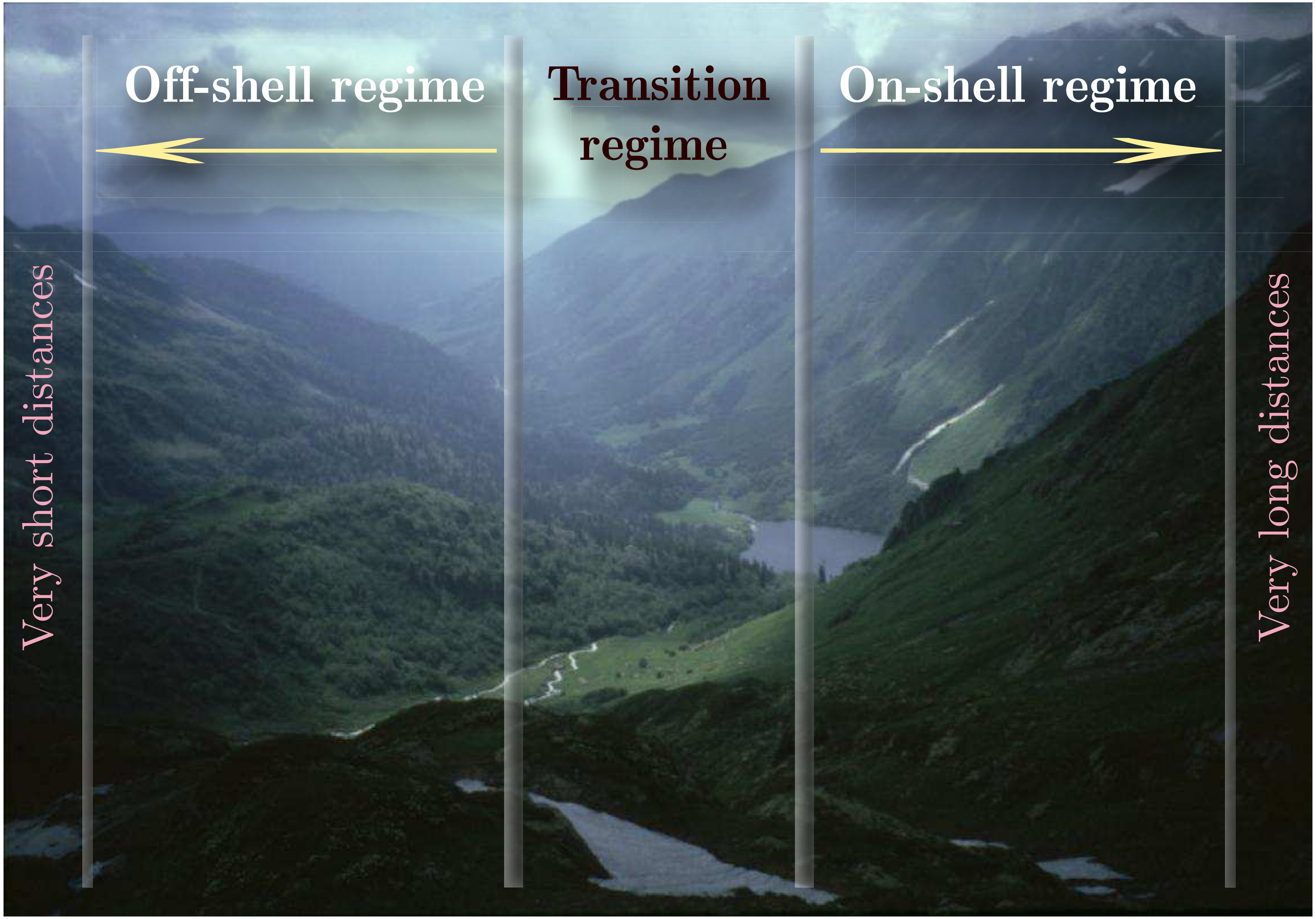
$$\begin{aligned} \mathbb{V}_{s,d}(q) &= \int dx e^{\pm iqx} \Big[\prod_{a \in I_s} e^{-ip_a x_a} \psi_a \left(\mathbf{p}_a, x_a - x\right) \Big] \Big[\prod_{b \in F_s} e^{ip_b x_b} \psi_b^* \left(\mathbf{p}_b, x_b - x\right) \Big], \\ &= (2\pi)^4 \widetilde{\delta}_{s,d} \left(q \mp q_{s,d}\right) \exp\left[-\mathfrak{S}_{s,d} \pm i \left(q \mp q_{s,d}\right) \cdot X_{s,d}\right] \end{aligned}$$

[the last equality is written in CRGP approximation]; $\mathcal{J}_{s,d}$ are the hadronic currents; Δ^j and Δ_μ^ν are the propagators of, respectively, the massive neutrino ν_j and W boson:

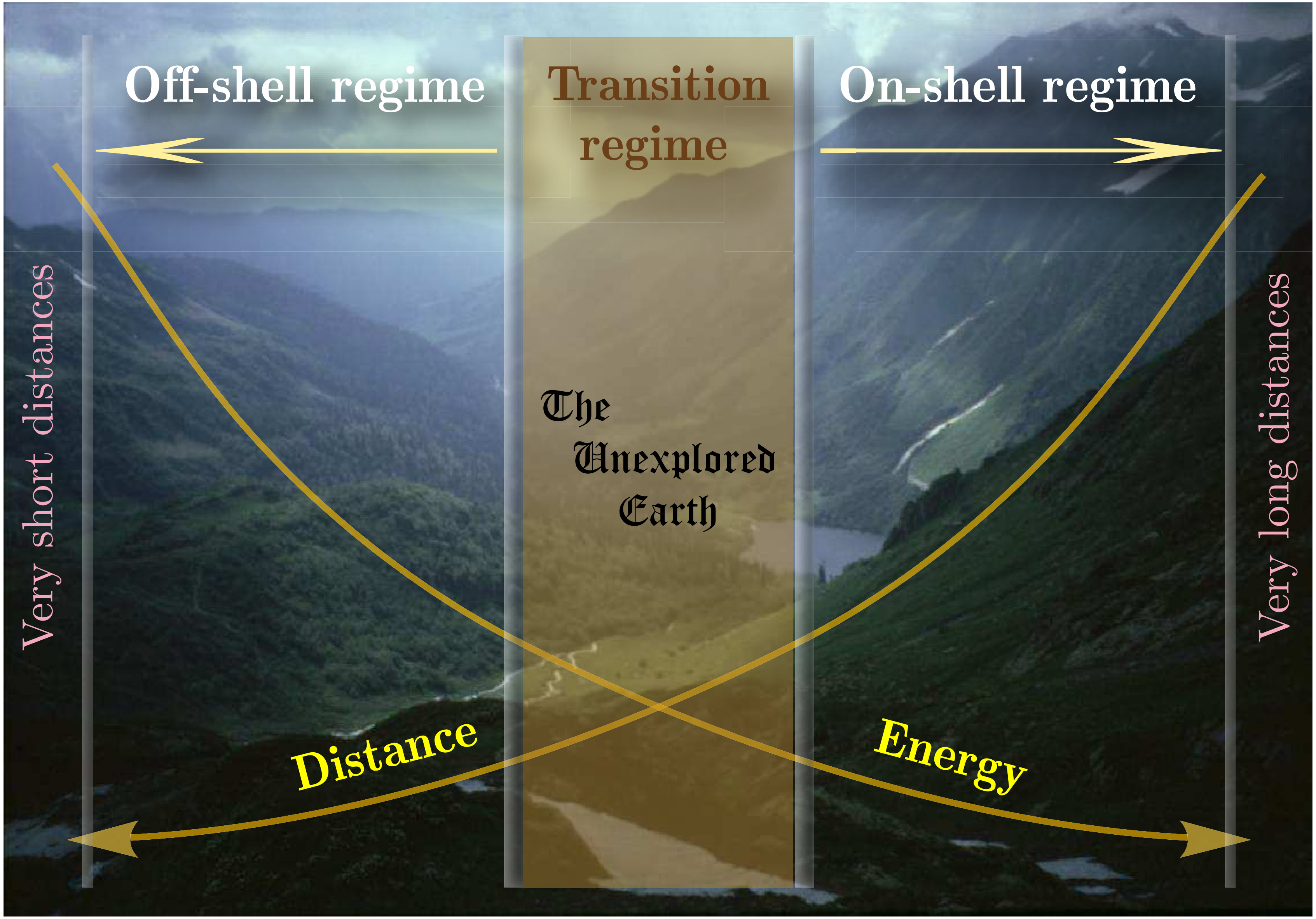
$$\Delta^j(q) = \frac{i}{\hat{q} - m_j + i0} = i \frac{\hat{q} + m_j}{q^2 - m_j^2 + i0}, \quad \text{etc.}$$

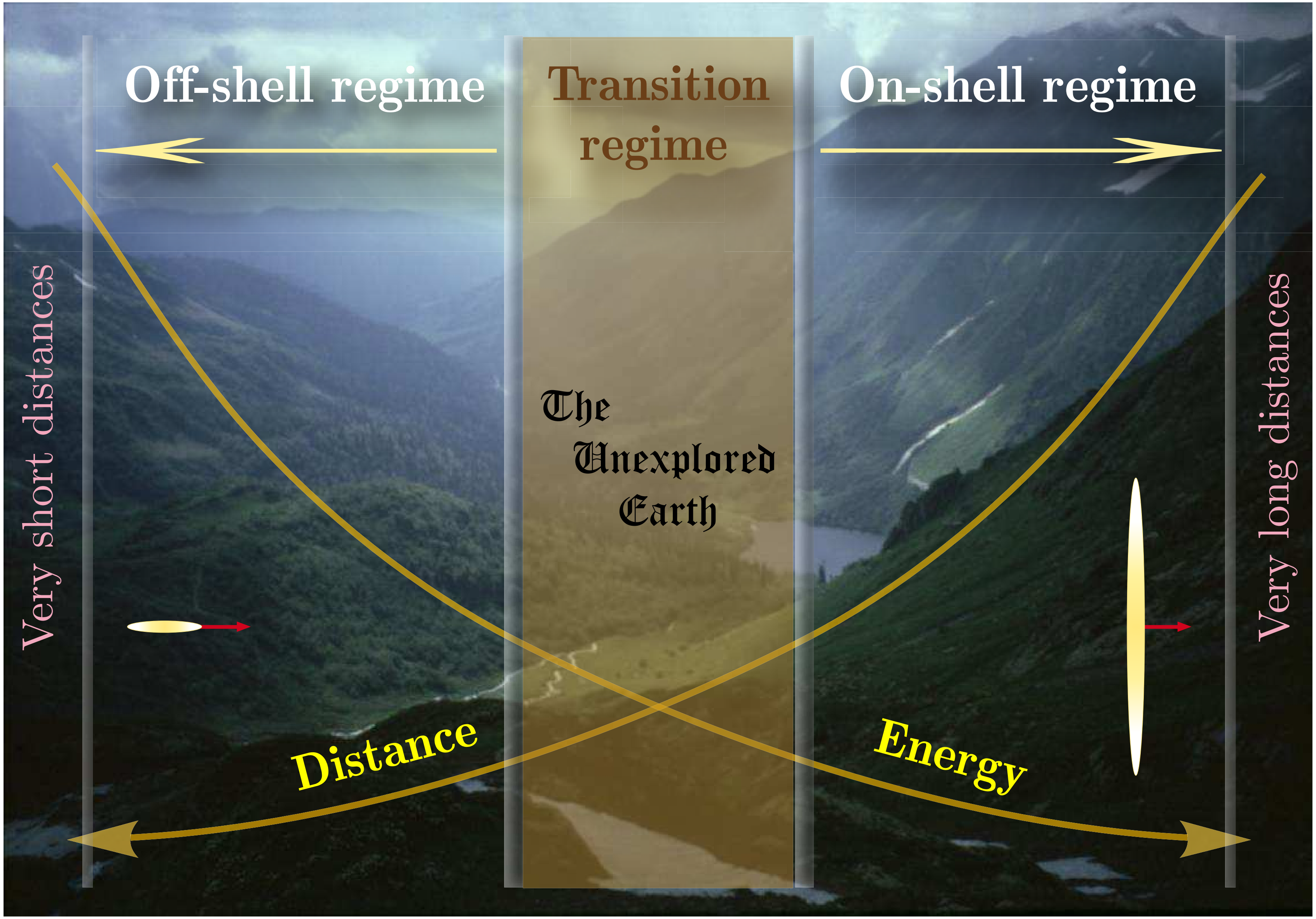
The bare W boson propagator has the form $\Delta_{\mu\nu}^{(\mathrm{b})}(k) = -i \dfrac{g_{\mu\nu} - k_\mu k_\nu / m_W^2}{k^2 - m_W^2 + i0}$. However, the explicit form of $\Delta_{\mu\nu}$ is not used below. So the latter can be thought of as the exact *renormalized* propagator.

The main problem is in calculation of the integral (6). Depending of the calculation method and corresponding assumptions one obtains several regimes in the behaviour of the amplitude.


Off-shell regime
Transition regime
On-shell regime
Very short distances
Very long distances


Off-shell regime
Transition regime
On-shell regime
Very short distances
The Unexplored Earth
Very long distances


Off-shell regime
Transition regime
On-shell regime
Very short distances
Very long distances
The Unexplored Earth
Distance
Energy


Off-shell regime
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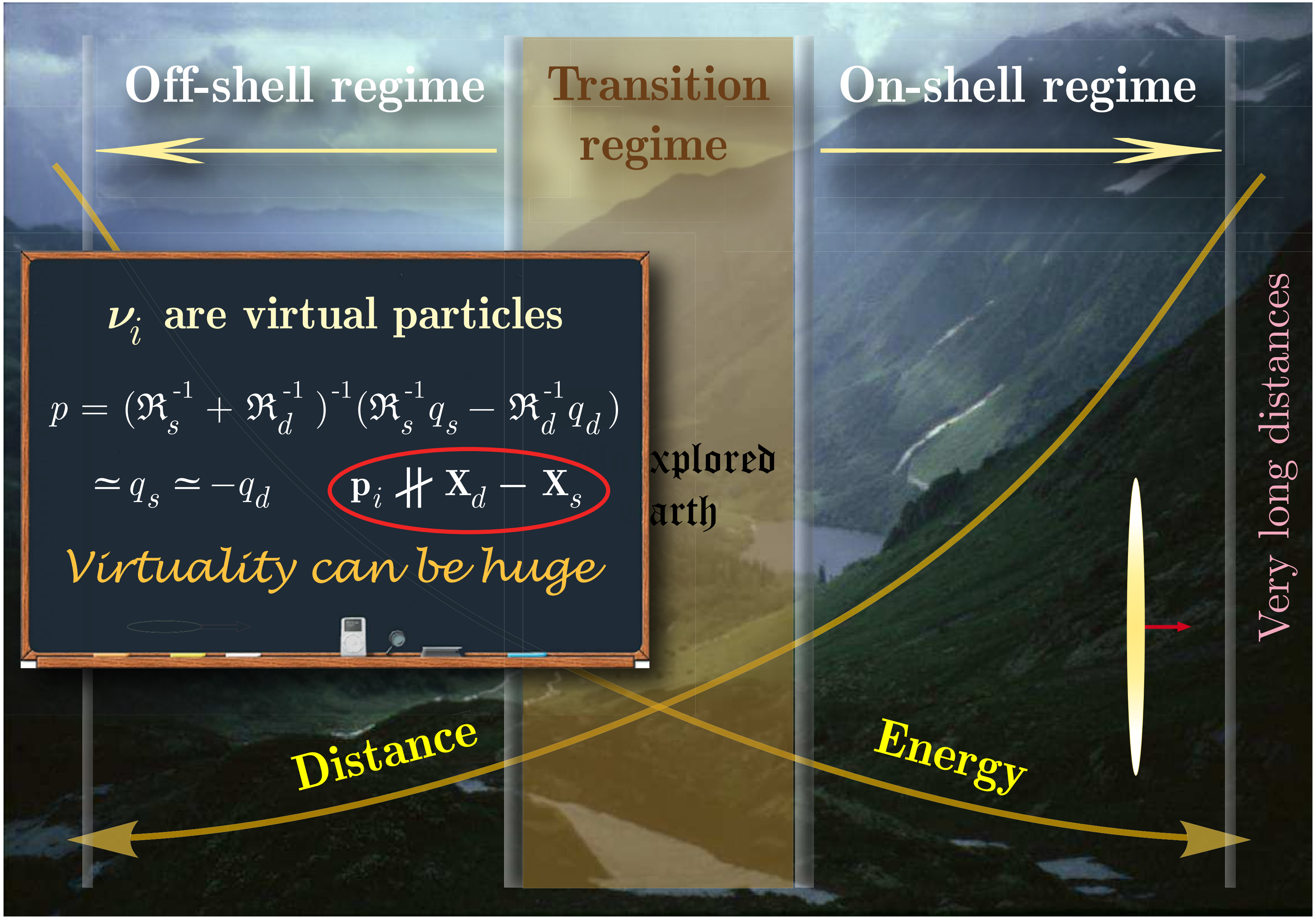
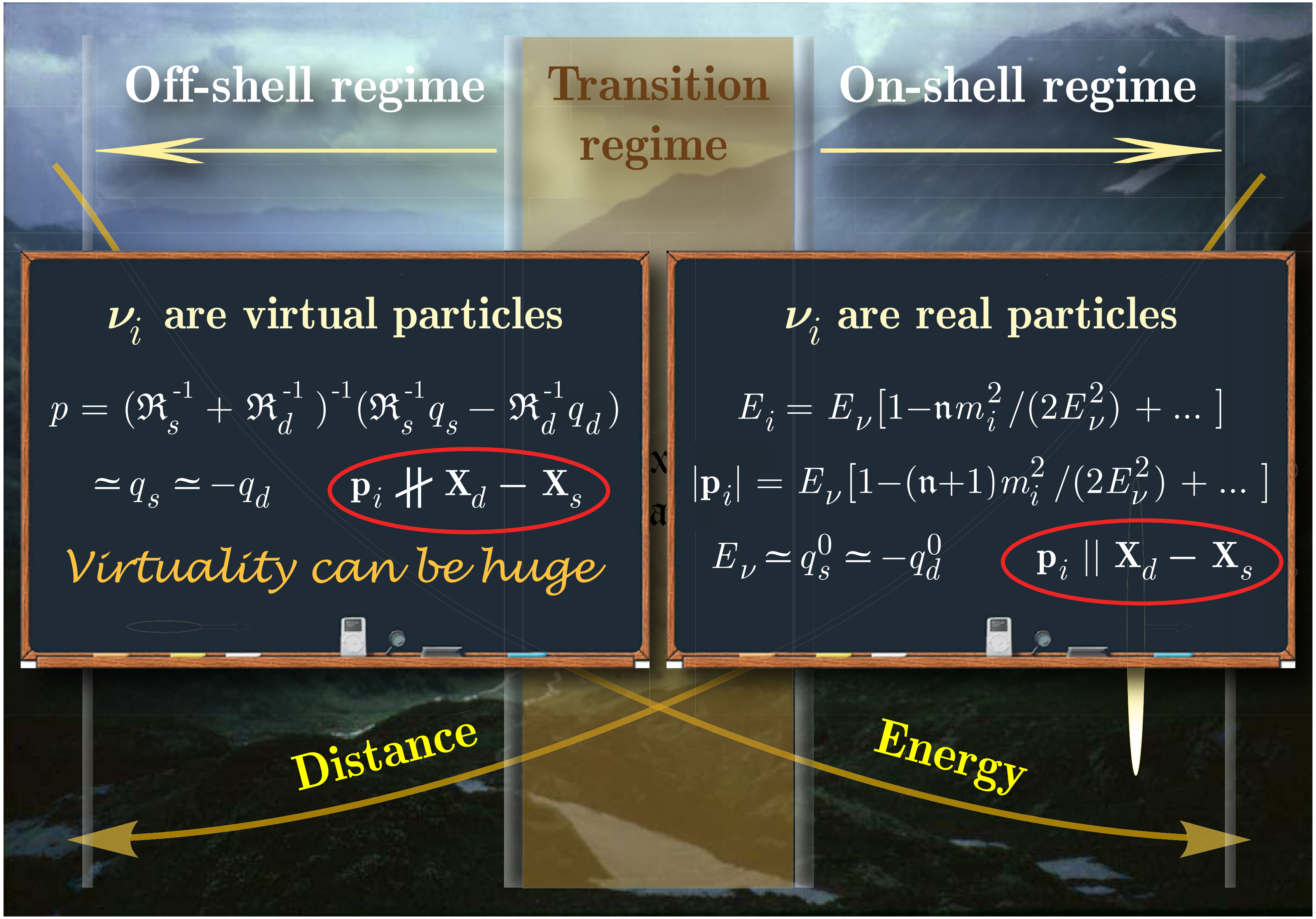
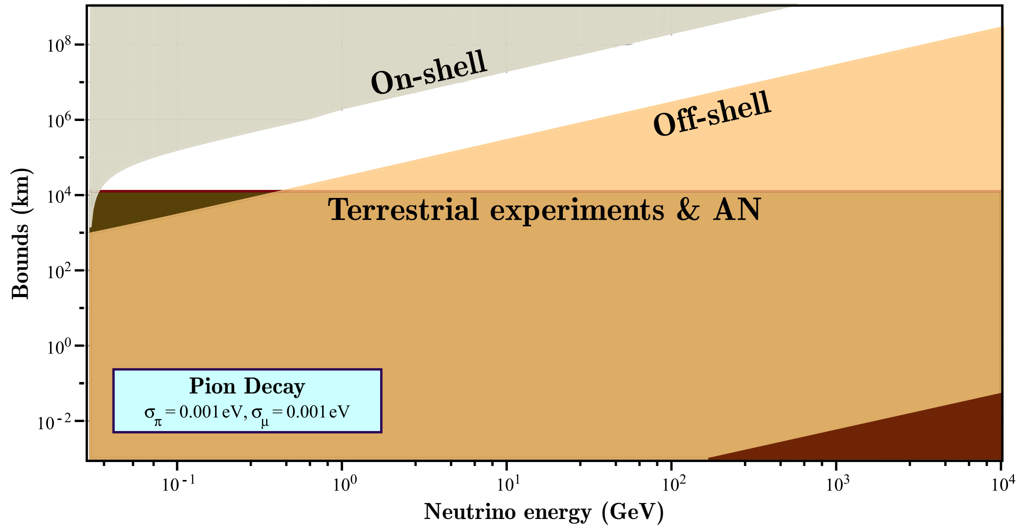
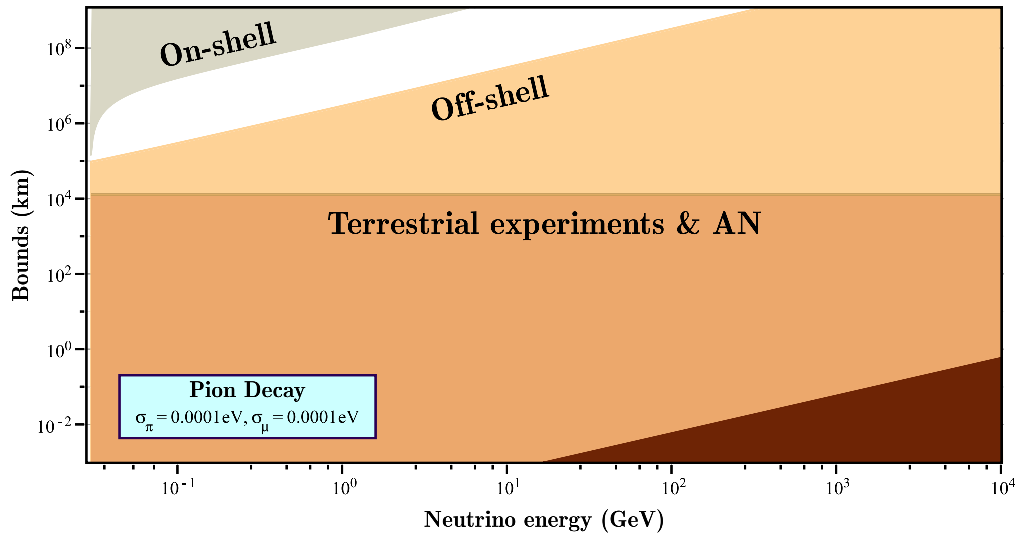
The Unexplored Earth
Very long distances
Distance
Energy


Off-shell regime
Transition regime
On-shell regime
ν_i are virtual particles
$p = (\mathfrak{R}_s^{-1} + \mathfrak{R}_d^{-1})^{-1}(\mathfrak{R}_s^{-1} q_s - \mathfrak{R}_d^{-1} q_d)$
$\simeq q_s \simeq -q_d$
$\mathbf{p}_i \nparallel \mathbf{X}_d - \mathbf{X}_s$
Virtuality can be huge
xplored
arth
Very long distances
Distance
Energy

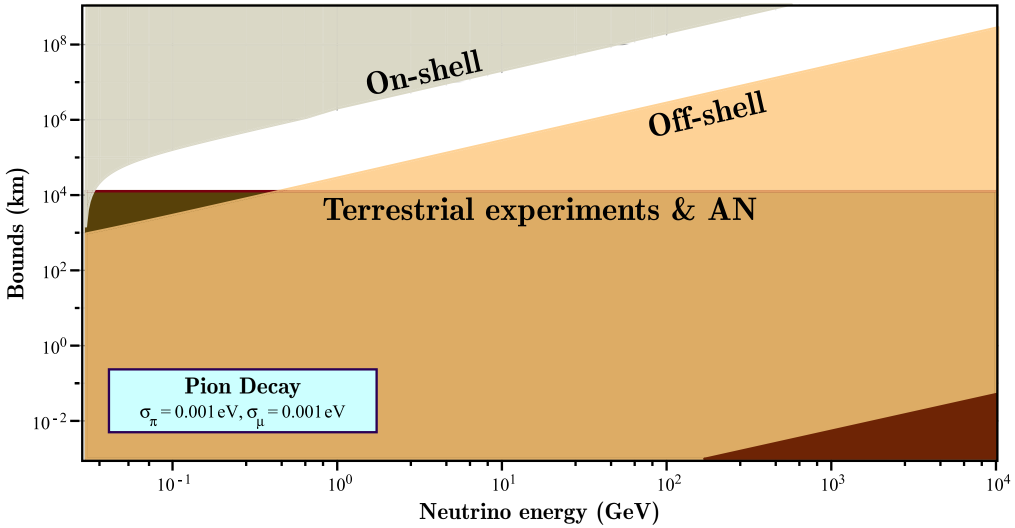


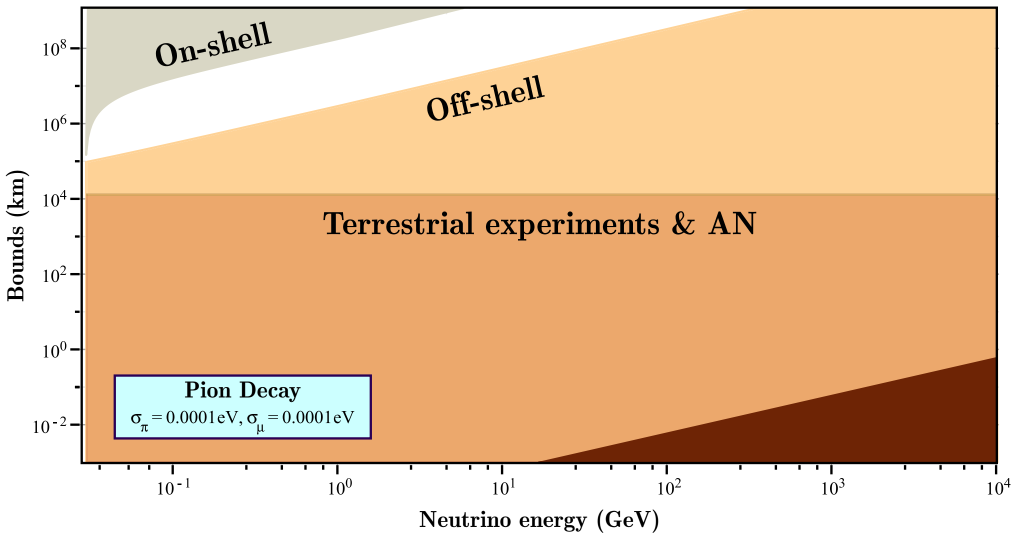
Off-shell regime
Transition regime
On-shell regime
ν_i are virtual particles
$p = (\mathfrak{R}_s^{-1} + \mathfrak{R}_d^{-1})^{-1}(\mathfrak{R}_s^{-1} q_s - \mathfrak{R}_d^{-1} q_d)$
$\simeq q_s \simeq -q_d$
$\mathbf{p}_i \nparallel \mathbf{X}_d - \mathbf{X}_s$
Virtuality can be huge
ν_i are real particles
$E_i = E_\nu [1 - \mathfrak{n} m_i^2/(2E_\nu^2) + \dots]$
$|\mathbf{p}_i| = E_\nu [1 - (\mathfrak{n}+1) m_i^2/(2E_\nu^2) + \dots]$
$E_\nu \simeq q_s^0 \simeq -q_d^0$
$\mathbf{p}_i \parallel \mathbf{X}_d - \mathbf{X}_s$
Distance
Energy





The “on-shell regime” does not work for the accelerator neutrino experiments and experiments with AN, while the range of applicability of the “off-shell regime” vell fits the $L-E$ range of these experiments.





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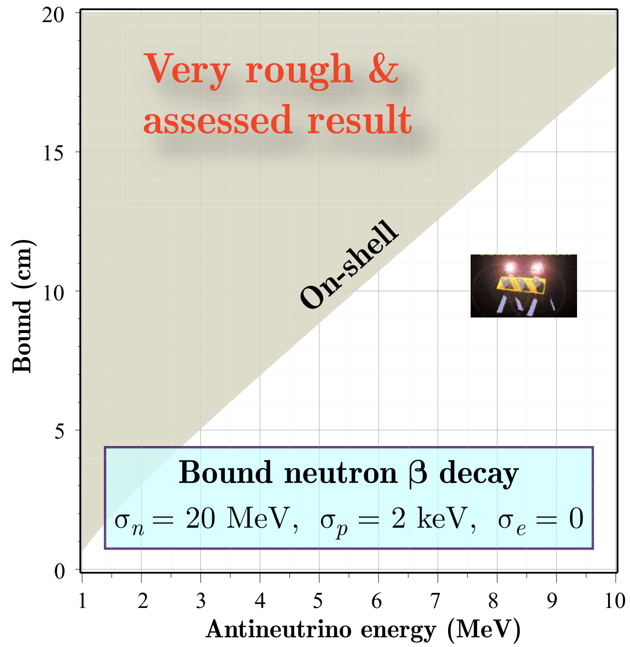


But (very preliminary conclusion!)

for the experiments with reactor & geophysical $\overline{\nu}_e$s the situation seems to be quite opposite.

Microscopic probability

In both on-shell and off-shell regimes the factorized and squared amplitude $|\mathcal{A}_{\beta\alpha}|^2$ for the standard light neutrinos can finally be represented in the following form:

$$
\begin{aligned}
|\mathcal{A}_{\beta\alpha}|^2 = & \frac{(2\pi)^4\delta_s(p-q_s)\mathrm{V}_s}{\prod_{\varkappa\in S} 2E_\varkappa \mathrm{V}_\varkappa} \frac{(2\pi)^4\delta_d(p+q_d)\mathrm{V}_d}{\prod_{\varkappa\in D} 2E_\varkappa \mathrm{V}_\varkappa} \\
& \times \left| M_s^- M_d^{-*} + M_s^+ M_d^{+*} \right|^2 \\
& \times \mathfrak{N} \left| \sum_j V_{\alpha j}^* V_{\beta j}\, e^{-\Omega_j} \right|^2 .
\end{aligned}
\tag{7}
$$

Here $M_{s,d}^{\pm}$ are the matrix elements of the production and detection of the left/right polarized neutrinos in the corresponding subprocesses:

$$M_s^{\pm} = \frac{g^2}{8}\overline{u}_{\pm}(\mathbf{p})\mathcal{J}_s^{\mu}\Delta_{\mu\nu}(p+p_\alpha)O^{\nu}v(\mathbf{p}_\alpha) \quad \text{[ME of } I_s \to F_s' + \ell_\alpha^+ + \nu_\pm\text{]},$$

$$M_d^{\pm *} = \frac{g^2}{8}\overline{u}(\mathbf{p}_\beta)O^{\nu}\Delta_{\nu\mu}(p-p_\beta)\mathcal{J}_d^{*\mu}u_{\pm}(\mathbf{p}) \quad \text{[ME of } \nu_\pm + I_d \to F_d' + \ell_\beta^-\text{]}.$$

Clearly for usual reactions the contribution with $M_s^+ M_d^{+*}$ in (7) and neutrino masses in the terms $M_s^- M_d^{-*}$ can safely be neglected. In other words, neutrinos in the matrix elements M_s^- and M_d^{-*} can be treated as real massless neutrinos with $p^2 = 0$.

Other common for on-shell and off-shell regimes ingredients in (7) are:

- $E_\varkappa = \sqrt{\mathbf{p}_\varkappa}$ and $\mathrm{V}_\varkappa$ are, respectively, the mean energies and (small) effective volumes of the packets $\varkappa$.
- $\delta_{s,d}$ are the "smeared" δ functions – analogous (but not identical) to the functions $\widetilde{\delta}_{s,d}$ involved into the amplitude.

  Responsible for the approximate conservation of the energy and momentum.

$$\delta_{s,d}(K) = \int \frac{dx}{(2\pi)^4} \exp\left(iKx - \frac{1}{2}\Re^{\mu\nu}_{s,d} x_\mu x_\nu\right) = \frac{\exp\left(-\frac{1}{2}\widetilde{\Re}^{\mu\nu}_{s,d} K_\mu K_\nu\right)}{(2\pi)^2\sqrt{|\Re_{s,d}|}},$$

- $\mathrm{V}_{s,d}$ are the effective 4D overlap volumes of the external packets in the source and detector,

$$\mathrm{V}_{s,d} = \int dx \prod_{\varkappa \in S,D} \left|\psi_\varkappa\left(\boldsymbol{p}_\varkappa, x_\varkappa - x\right)\right|^2 = \frac{\pi^2 \exp\left(-2\mathfrak{S}_{s,d}\right)}{4\sqrt{|\Re_{s,d}|}}.$$

  Responsible for the "genetic selection" between the lucky and unlucky configuration of the world tubes of the wave packets.

The distinction between the two regimes is quite essential:

$$\mathfrak{N} \propto \begin{cases} \dfrac{\Re_{\mu\nu} l^\mu l^\nu}{|\mathbf{X}|^2} \left(1 + \text{corrections}\right) & [\text{very slowly depend on } p_j] \text{ on-shell}, \\ \dfrac{\det \Re}{\Re_{\mu\nu} p_j^\mu p_j^\nu} \left(1 + \text{corrections}\right) & [\text{very slowly depend on } X\ ] \text{ off-shell}, \end{cases}$$

$$\Omega_j = \begin{cases} i(p_j X) + \dfrac{\left[(p_j X)^2 - m_j^2 X^2\right]}{\Re_{\mu\nu} p_j^\mu p_j^\nu} + \text{corrections} & \text{on-shell}, \\ i(p_j X) + \dfrac{\left(p p_j - m_j^2\right)^2}{4\Re_{\mu\nu} p_j^\mu p_j^\nu} + \mathfrak{T}_j^{\mu\nu} X_\mu X_\nu + \text{corrections} & \text{off-shell}. \end{cases}$$

Here

$$\mathfrak{T}_j^{\mu\nu} = \frac{\left(\mathfrak{R}_{\mu\nu}\mathfrak{R}_{\lambda\rho} - \mathfrak{R}_{\mu\lambda}\mathfrak{R}_{\nu\rho}\right) p_j^\lambda p_j^\rho}{\mathfrak{R}_{\mu\nu} p_j^\mu p_j^\nu},$$

$$l = (1, \mathbf{l}), \ \ \mathbf{l} = \frac{\mathbf{X}}{|\mathbf{X}|}, \quad X = X_s - X_d = (X_0, \mathbf{X}).$$

As is discussed below, the 4-momenta p_j in the two regimes represent rather different mathematical constructions, which have very different physical meaning.

- It the on-shell UR regime, the components of p_j are given by the series in powers of the small dimensionless parameter

$$r_j = \frac{m_j^2}{2E_\nu^2},$$

namely,

$$p_j^0 = E_\nu\left(1 - \sum_{n=1}^{\infty} C_n^E r_j^n\right), \quad |\mathbf{p}_j| = E_\nu\left(1 - \sum_{n=1}^{\infty} C_n^P r_j^n\right), \quad \mathbf{p}_j = |\mathbf{p}_j|\mathbf{l}, \quad p^2 = m_j^2.$$

where

$$E_\nu = \frac{(Yl)}{\Re_{\mu\nu}l^\mu l^\nu}, \quad Y^\mu = \widetilde{\Re}_s^{\mu\nu}q_{s\nu} - \widetilde{\Re}_d^{\mu\nu}q_{d\nu}, \quad \Re^{\mu\nu} = \Re_s^{\mu\nu} + \Re_d^{\mu\nu}$$

[for any tensor A, $\widetilde{A} = gA^{-1}g$.]

- It the off-shell UR regime p_j is not the effective neutrino 4-momentum, but only a notation:

$$p_j = (p_j^0, \mathbf{p}), \quad p_j^0 \equiv E_j = \sqrt{|\mathbf{p}|^2 + m_j^2},$$

while the "intrinsic" 4-momentum is

$$p = (p_0, \mathbf{p}), \quad p^\mu = \Re^{\mu\nu}Y_\nu \approx q_s \approx -q_d \quad \Longrightarrow \quad p^2 \neq m_j^2.$$

Moreover, there is no a straight connection between the directions of $\mathbf{p}$ and $\mathbf{l}$.

Neutrino Virtuality

In the Off-shell regime the allowed neutrino virtuality is defined by the condition

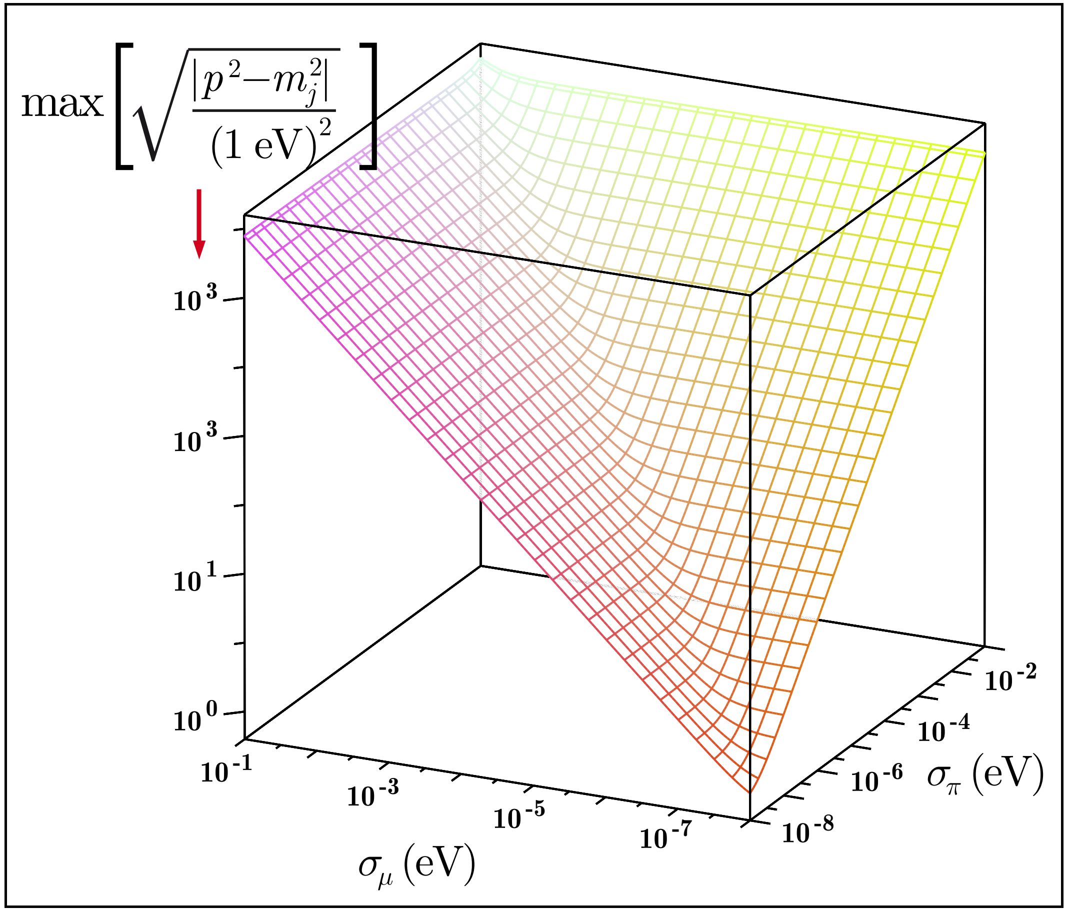
$$\left|p^2 - m_j^2\right| \lesssim 2\sqrt{\mathcal{G}_j}$$
$$(j = 1, 2, 3, \ldots),$$

where $\mathcal{G}_j$ is generally a Lorentz invariant function of the external momenta.

For neutrinos from $\pi_{\mu 2}$-decay

$$\mathcal{G}_j = \left(\sigma_\pi^2 m_\mu^2 + \sigma_\mu^2 m_\pi^2\right)\left|\mathbf{u}_\mu^\star\right|^2.$$

So $\left|p^2 - m_j^2\right|$ can be large comparing to m_j^2.



The virtuality is not itself an observable quantity since the virtual neutrinos contribute to $\langle\langle |A|^2 \rangle\rangle$ (the squared amplitude averaged over the unmeasured external momenta) exactly as the normal on-shell particles. However certain "footprints" of the virtuality must remain in the corrections to the survival and transition probabilities. In particular, the transitions "light neutrinos $\longleftrightarrow$ heavy neutrinos should be strongly suppressed.

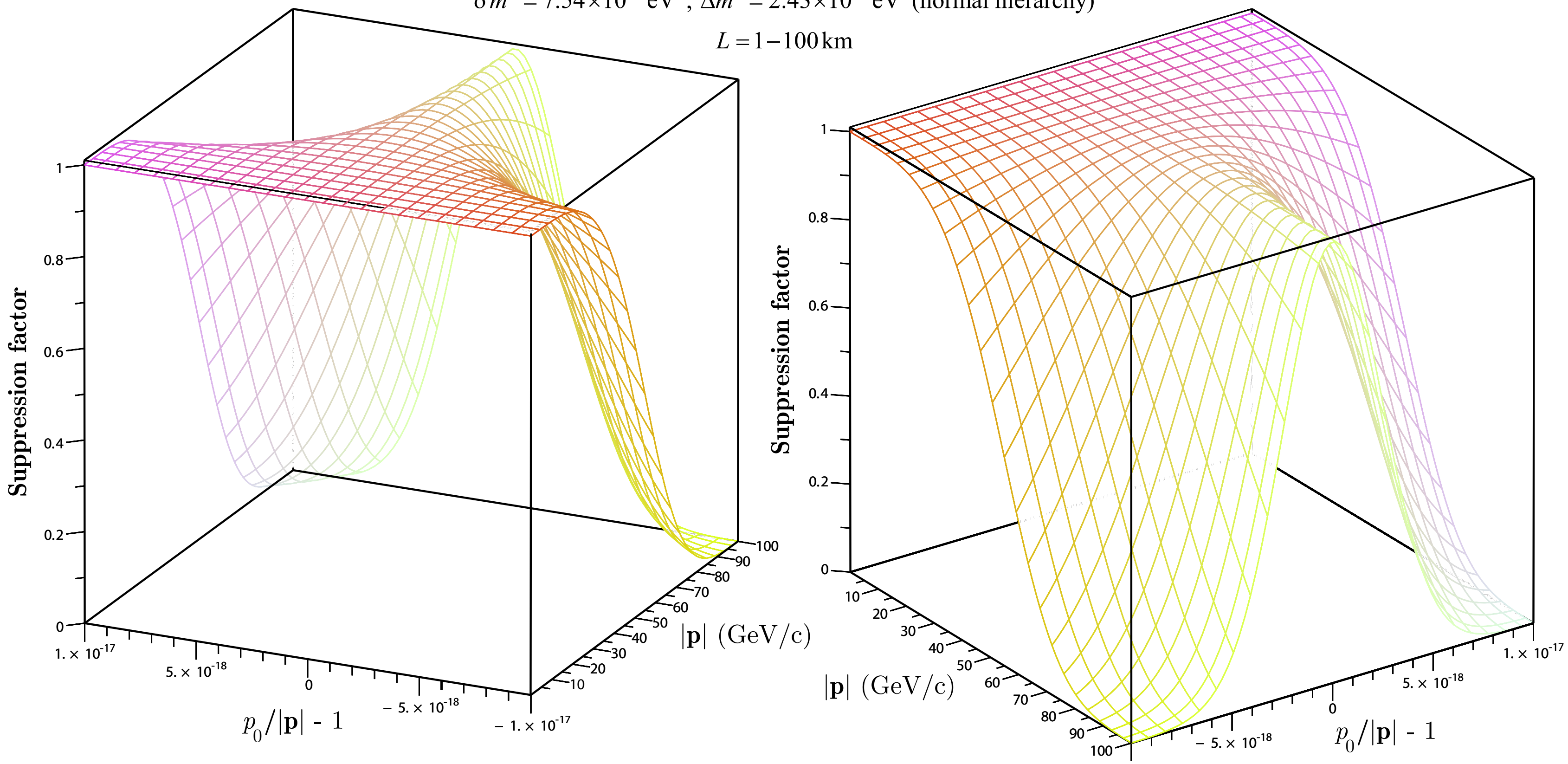
An effect of the neutrino virtuality

$\sigma_\pi = \sigma_\mu = 5\times10^{-4}\ \text{eV}$

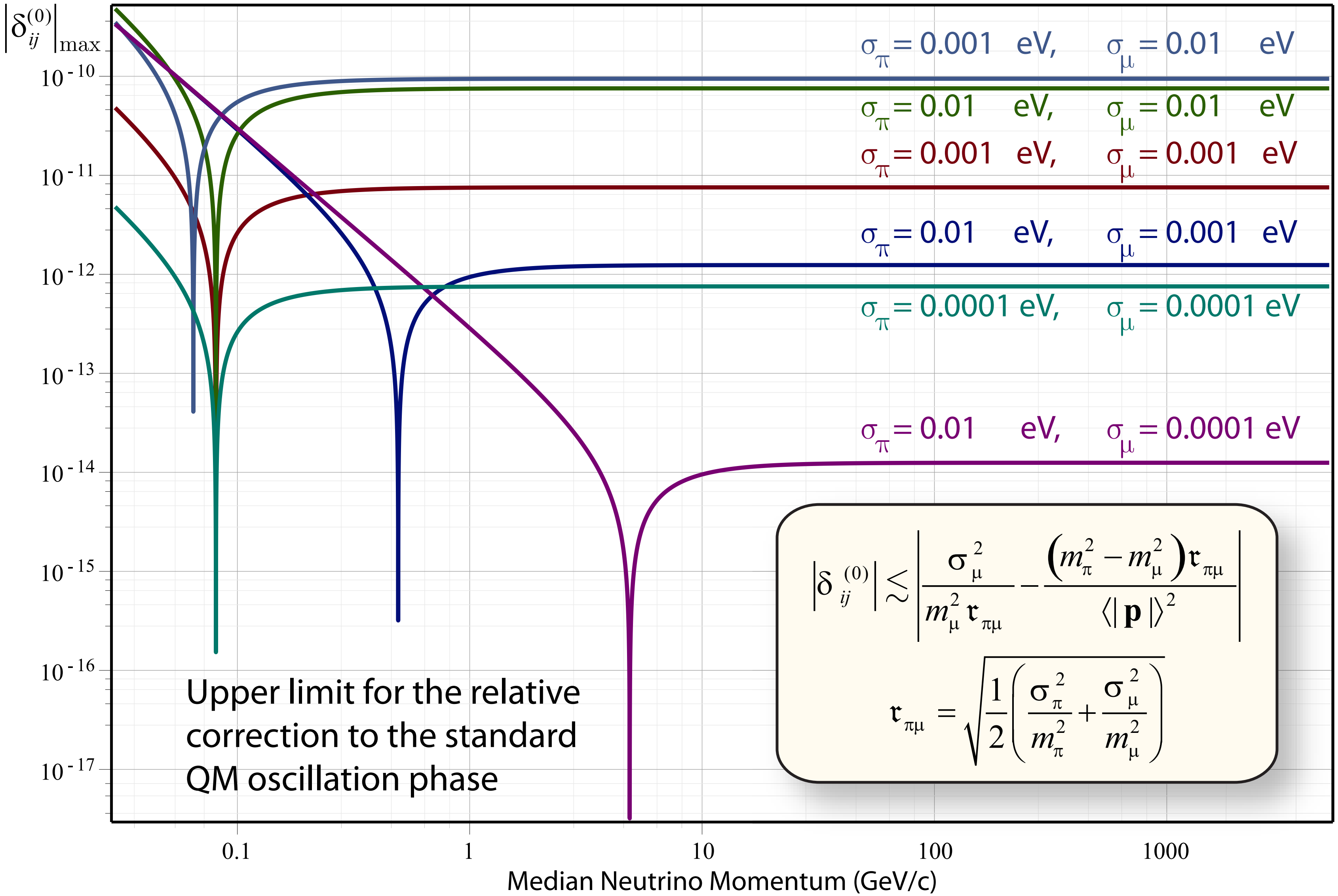
$s_{12}^2 = 0.307,\ s_{23}^2 = 0.386,\ s_{13}^2 = 0.0241,\ \delta_{\text{CP}} = 1.08\pi,$

$\delta m^2 = 7.54\times10^{-5}\ \text{eV}^2,\ \Delta m^2 = 2.43\times10^{-3}\ \text{eV}^2$ (normal hierarchy)

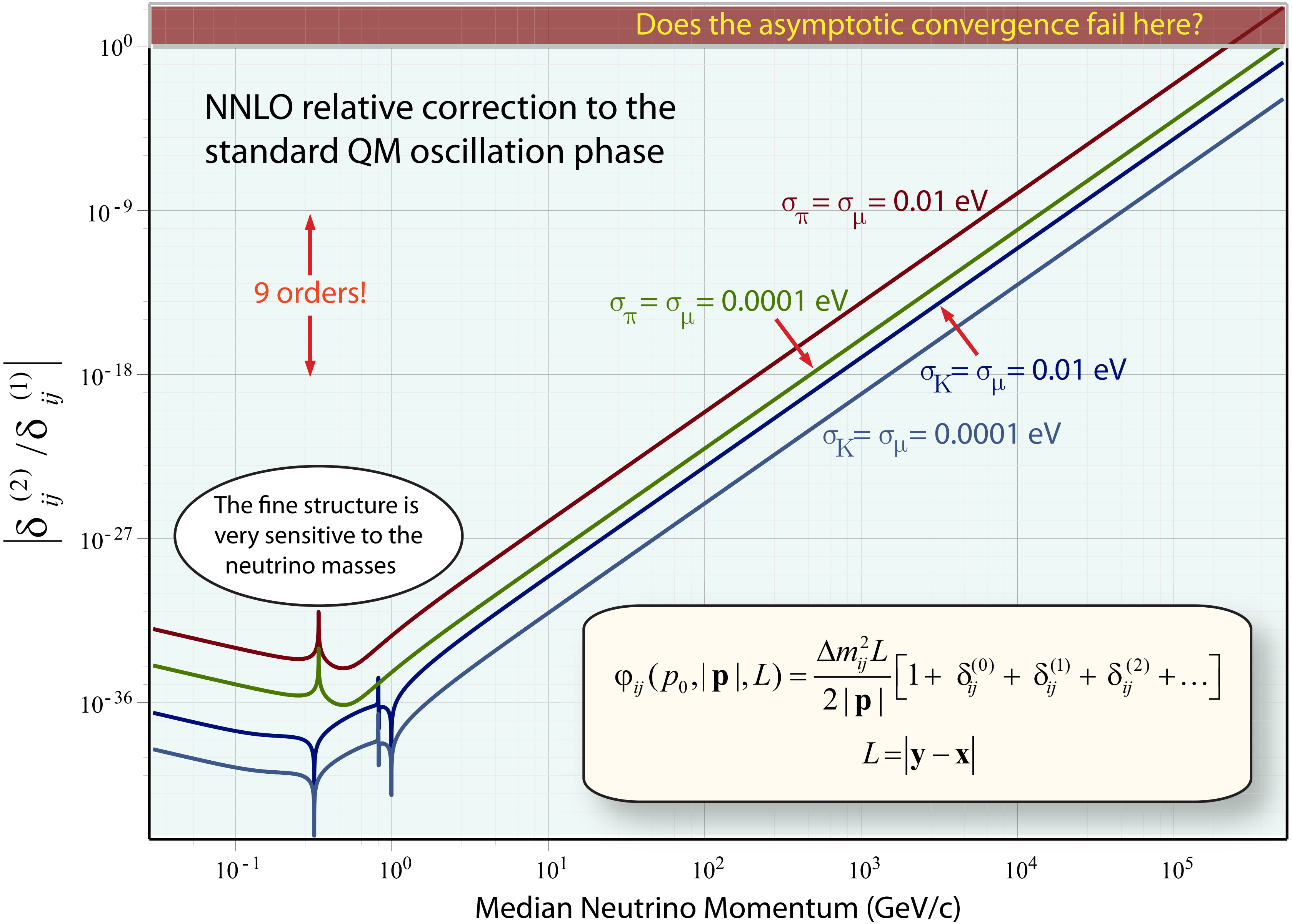
$L = 1{-}100\,\text{km}$



The real scale of the effect is not yet well understood. More studies are needed.



$|\delta_{ij}^{(0)}|_{\max}$
10^{-10}
10^{-11}
10^{-12}
10^{-13}
10^{-14}
10^{-15}
10^{-16}
10^{-17}
0.1
1
10
100
1000
Median Neutrino Momentum (GeV/c)
$\sigma_\pi = 0.001$ eV, $\sigma_\mu = 0.01$ eV
$\sigma_\pi = 0.01$ eV, $\sigma_\mu = 0.01$ eV
$\sigma_\pi = 0.001$ eV, $\sigma_\mu = 0.001$ eV
$\sigma_\pi = 0.01$ eV, $\sigma_\mu = 0.001$ eV
$\sigma_\pi = 0.0001$ eV, $\sigma_\mu = 0.0001$ eV
$\sigma_\pi = 0.01$ eV, $\sigma_\mu = 0.0001$ eV
Upper limit for the relative correction to the standard QM oscillation phase
$\left|\delta_{ij}^{(0)}\right| \lesssim \left| \frac{\sigma_\mu^2}{m_\mu^2 \mathfrak{r}_{\pi\mu}} - \frac{(m_\pi^2 - m_\mu^2)\mathfrak{r}_{\pi\mu}}{\langle|\mathbf{p}|\rangle^2} \right|$
$\mathfrak{r}_{\pi\mu} = \sqrt{\frac{1}{2}\left(\frac{\sigma_\pi^2}{m_\pi^2} + \frac{\sigma_\mu^2}{m_\mu^2}\right)}$



Does the asymptotic convergence fail here?
NNLO relative correction to the standard QM oscillation phase
9 orders!
$\sigma_\pi = \sigma_\mu = 0.01$ eV
$\sigma_\pi = \sigma_\mu = 0.0001$ eV
$\sigma_K = \sigma_\mu = 0.01$ eV
$\sigma_K = \sigma_\mu = 0.0001$ eV
The fine structure is very sensitive to the neutrino masses
$\varphi_{ij}(p_0, |\mathbf{p}|, L) = \frac{\Delta m_{ij}^2 L}{2|\mathbf{p}|}\left[1 + \delta_{ij}^{(0)} + \delta_{ij}^{(1)} + \delta_{ij}^{(2)} + \ldots\right]$
$L = |\mathbf{y} - \mathbf{x}|$
$\left|\delta_{ij}^{(2)} / \delta_{ij}^{(1)}\right|$
10^0
10^{-9}
10^{-18}
10^{-27}
10^{-36}
10^{-1}
10^0
10^1
10^2
10^3
10^4
10^5
Median Neutrino Momentum (GeV/c)

Macroscopic averaging for the on-shell case

To obtain the observable quantities, the probability must be averaged/integrated over all the unmeasurable or unused variables of incoming/outgoing WP states.

Such a procedure can only be realized by taking into account the conditions of a real experimental environment. For these reasons and in this sense, further analysis is model-dependent.

A thought experiment:

Assume that the statistical distributions of the incoming WPs $a \in I_{s,d}$ over the mean momenta, spin projections, and space-time coordinates in the source and detector "devices" can be described by the one-particle distribution functions $f_a(\boldsymbol{p}_a, s_a, x_a)$. It is convenient to normalize each function f_a to the total number, $N_a(x_a^0)$, of the packets a at a time x_a^0:

$$\sum_{s_a} \int \frac{d\boldsymbol{x}_a d\boldsymbol{p}_a}{(2\pi)^3} f_a(\boldsymbol{p}_a, s_a, x_a) = N_a(x_a^0) \qquad (a \in I_{s,d}).$$

For clarity purposes, we (re)define the terms "source" and "detector":

$$\mathcal{S} = \text{supp}_{\{x_a;\ a \in I_s\}} \prod_a f_a(\boldsymbol{p}_a, s_a, x_a), \quad \mathcal{D} = \text{supp}_{\{x_a;\ a \in I_d\}} \prod_a f_a(\boldsymbol{p}_a, s_a, x_a).$$

We'll use the same terms and notation $\mathcal{S}$ and $\mathcal{D}$ also for the corresponding devices.

Suppositions:

[1] $\mathcal{S}$ and $\mathcal{D}$ are finite and mutually disjoint within the space domain.

[2] Effective spatial dimensions of $\mathcal{S}$ and $\mathcal{D}$ are small compared to the mean distance between them but very large compared to the effective dimensions ($\sim \sigma_{\varkappa}^{-1}$) of all WPs in $\mathcal{S}$ and $\mathcal{D}$.

[3] The experiment measures only the momenta of the secondaries in $\mathcal{D}$ and (due to [2]) the background events caused by the secondaries falling into $\mathcal{D}$ from $\mathcal{S}$ can be neglected.

[4] The detection efficiency in $\mathcal{D}$ is 100%.

With these assumptions, the macroscopically averaged probability represents the total number, $dN_{\alpha\beta}$, of the events recorded in $\mathcal{D}$ and consisted of the secondaries $b \in F_d$ having the mean momenta between $\boldsymbol{p}_b$ and $\boldsymbol{p}_b + d\boldsymbol{p}_b$:

$$\begin{aligned}\langle\langle|\mathcal{A}_{\beta\alpha}|^2\rangle\rangle \equiv dN_{\alpha\beta} = & \sum_{\text{spins}} \int \prod_{a \in I_s} \frac{d\boldsymbol{x}_a d\boldsymbol{p}_a f_a(\boldsymbol{p}_a, s_a, x_a)}{(2\pi)^3 2E_a \mathrm{V}_a} \int \prod_{b \in F_s} \frac{d\boldsymbol{x}_b d\boldsymbol{p}_b}{(2\pi)^3 2E_b \mathrm{V}_b} \mathrm{V}_s \\ & \times \int \prod_{a \in I_d} \frac{d\boldsymbol{x}_a d\boldsymbol{p}_a f_a(\boldsymbol{p}_a, s_a, x_a)}{(2\pi)^3 2E_a \mathrm{V}_a} \int \prod_{b \in F_d} \frac{d\boldsymbol{x}_b [d\boldsymbol{p}_b]}{(2\pi)^3 2E_b \mathrm{V}_b} \mathrm{V}_d \\ & \times \int dE_\nu (2\pi)^4 \delta_s(p_\nu - q_s) |M_s|^2 (2\pi)^4 \delta_d(p_\nu + q_d) |M_d|^2 \\ & \times \frac{\mathfrak{D}}{2\sqrt{\pi}(2\pi)^3 L^2} \left| \sum_j V_{\alpha j}^* V_{\beta j}\, e^{-\Omega_j(T,L) - \Theta_j} \right|^2. \end{aligned} \qquad (8)$$

- $\sum_{\text{spins}}$ denotes the averaging/summation over the spin projections of the in/out states.
- Symbol $[d\boldsymbol{p}_b]$ indicates that integration in variable $\boldsymbol{p}_b$ is not performed, i.e., $\int [d\boldsymbol{p}_b] = d\boldsymbol{p}_b$.

Under additional assumptions, the unwieldy expression (8) can be simplified in a few steps.

Step 1: Multidimensional integration in WP positions.

Supposition 5: The distribution functions $f_a(\boldsymbol{p}_a, s_a, x_a)$, as well as the factors $e^{-\Omega_j-\Omega_i^*}/L^2$ vary at large (macroscopic) scales.

> The integrand $\prod_\varkappa |\psi_\varkappa(\boldsymbol{p}_\varkappa, x_\varkappa - x)|^2$ in the integral representation of the overlap volumes (**??**) is essentially different from zero only if the classical word lines of all packets $\varkappa$ pass through a small (though not necessarily microscopic) vicinity of the integration variable.

Supposition 6: The edge effects can be neglected (a harmless extension of supposition [2]).

As a result, expression (8) is reduced to the following:

$$dN_{\alpha\beta} = \sum_{\text{spins}} \int dx \int dy \int d\mathfrak{P}_s \int d\mathfrak{P}_d \int dE_\nu \frac{\mathfrak{D} \left| \sum_j V_{\alpha j}^* V_{\beta j} \, e^{-\Omega_j(T,L)-\Theta_j} \right|^2}{16\pi^{7/2}|\boldsymbol{y}-\boldsymbol{x}|^2}, \tag{9}$$

where $T = y_0 - x_0$, $L = |\boldsymbol{y} - \boldsymbol{x}|$ and we have defined the differential forms

$$d\mathfrak{P}_s = \prod_{a\in I_s} \frac{d\boldsymbol{p}_a f_a(\boldsymbol{p}_a, s_a, x)}{(2\pi)^3 2E_a} \prod_{b\in F_s} \frac{d\boldsymbol{p}_b}{(2\pi)^3 2E_b} (2\pi)^4 \delta_s(p_\nu - q_s)|M_s|^2, \tag{10a}$$

$$d\mathfrak{P}_d = \prod_{a\in I_d} \frac{d\boldsymbol{p}_a f_a(\boldsymbol{p}_a, s_a, y)}{(2\pi)^3 2E_a} \prod_{b\in F_d} \frac{[d\boldsymbol{p}_b]}{(2\pi)^3 2E_b} (2\pi)^4 \delta_d(p_\nu + q_d)|M_d|^2. \tag{10b}$$

Step 2: Integration in time variables.

Supposition 7: During the experiment, the distribution functions f_a in $\mathcal{S}$ and $\mathcal{D}$ vary slowly enough with time so that they can be modelled by the "rectangular ledges"

$$\begin{aligned} f_a(\boldsymbol{p}_a, s_a; x) &= \theta\left(x^0 - x_1^0\right)\theta\left(x_2^0 - x^0\right)\overline{f}_a(\boldsymbol{p}_a, s_a; \boldsymbol{x}) \ \text{ for } \ a \in I_s, \\ f_a(\boldsymbol{p}_a, s_a; y) &= \theta\left(y^0 - y_1^0\right)\theta\left(y_2^0 - y^0\right)\overline{f}_a(\boldsymbol{p}_a, s_a; \boldsymbol{y}) \ \text{ for } \ a \in I_d. \end{aligned} \tag{11}$$

Supposition 8: The time intervals needed to switch on and switch off the source and detector are negligibly small in comparison with periods of stationarity $\tau_s = x_2^0 - x_1^0$ and $\tau_d = y_2^0 - y_1^0$.

In case of detector, the step functions in (11) can be thought as the "hardware" or "software" trigger conditions. The periods of stationarity τ_s and τ_d can be astronomically long, as it is for the solar and atmospheric neutrino experiments ($\tau_s \ggg \tau_d$ in these cases), or very short, like in the experiments with short-pulsed accelerator beams (when usually $\tau_s \lesssim \tau_d$).

Within the model (11), the only time-dependent factor in the integrand of (9) is $e^{-\Omega_j - \Omega_i^*}$. So the problem is reduced to the (comparatively) simple integral

$$\int_{y_1^0}^{y_2^0} dy^0 \int_{x_1^0}^{x_2^0} dx^0 \, e^{-\Omega_j(y^0 - x^0, L) - \Omega_i^*(y^0 - x^0, L)} = \frac{\sqrt{\pi}}{2\mathfrak{D}} \tau_d \exp\left(i\varphi_{ij} - \mathscr{A}_{ij}^2\right) S_{ij}. \tag{12}$$

In relation (12) we have adopted the following notation:

$$S_{ij} = \frac{\exp\left(-\mathscr{B}_{ij}^2\right)}{4\tau_d\mathfrak{D}} \sum_{l,l'=1}^{2} (-1)^{l+l'+1} \mathsf{lerf}\left[2\mathfrak{D}\left(x_l^0 - y_{l'}^0 + \frac{L}{v_{ij}}\right) - i\mathscr{B}_{ij}\right], \tag{13}$$

$$\mathscr{A}_{ij} = (v_j - v_i)\mathfrak{D}L = \frac{2\pi\mathfrak{D}L}{E_\nu L_{ij}}, \qquad \mathscr{B}_{ij} = \frac{\Delta E_{ji}}{4\mathfrak{D}} = \frac{\pi\mathfrak{n}}{2\mathfrak{D}L_{ij}}, \tag{14}$$

$$\varphi_{ij} = \frac{2\pi L}{L_{ij}}, \qquad L_{ij} = \frac{4\pi E_\nu}{\Delta m_{ij}^2}, \qquad \frac{1}{v_{ij}} = \frac{1}{2}\left(\frac{1}{v_i} + \frac{1}{v_j}\right),$$

$$\Delta m_{ij}^2 = m_i^2 - m_j^2, \qquad \Delta E_{ij} = E_i - E_j,$$

$$\mathsf{lerf}(z) = \int_0^z dz' \mathsf{erf}(z') + \frac{1}{\sqrt{\pi}} = z\,\mathsf{erf}(z) + \frac{1}{\sqrt{\pi}} e^{-z^2},$$

For a more realistic description of the beam pulse experiments, the model (11) could be readily extended by inclusion of a series of rectangular ledges followed by pauses during which $f_a = 0$.

Then substituting (12) into (9) we obtain:

$$dN_{\alpha\beta} = \tau_d \sum_{\text{spins}} \int d\boldsymbol{x} \int d\boldsymbol{y} \int d\mathfrak{P}_s \int d\mathfrak{P}_d \int dE_\nu \frac{\mathcal{P}_{\alpha\beta}(E_\nu, |\boldsymbol{y}-\boldsymbol{x}|)}{4(2\pi)^3|\boldsymbol{y}-\boldsymbol{x}|^2}, \tag{15a}$$

$$\equiv \frac{\tau_d}{V_{\mathcal{D}}V_{\mathcal{S}}} \int d\boldsymbol{x} \int d\boldsymbol{y} \int d\Phi_\nu \int d\sigma_{\nu\mathcal{D}} \mathcal{P}_{\alpha\beta}(E_\nu, |\boldsymbol{y}-\boldsymbol{x}|). \tag{15b}$$

The differential forms $d\mathfrak{P}_{s,d}$ in (15a) are are given by eq. (10) after substitution $f_a \longmapsto \overline{f}_a$.

Explanation of the factors in eq. (15b).

- $V_{\mathcal{S}}$ and $V_{\mathcal{D}}$ are the spatial volumes of the source and detector, respectively.
- The differential form $d\varPhi_\nu$ is defined in such a way that the integral

$$\frac{d\boldsymbol{x}}{V_{\mathcal{S}}}\int\frac{d\varPhi_\nu}{dE_\nu} = d\boldsymbol{x}\sum_{\text{spins}\,\in\,S}\int\frac{d\mathfrak{P}_s E_\nu}{2(2\pi)^3|\boldsymbol{y}-\boldsymbol{x}|^2} \tag{16}$$

is the flux density of neutrinos in $\mathcal{D}$, produced through the processes $I_s \to F'_s \ell^+_\alpha \nu$ in $\mathcal{S}$.

More precisely, it is the number of neutrinos appearing per unit time and unit neutrino energy in an elementary volume $d\boldsymbol{x}$ around the point $\boldsymbol{x} \in \mathcal{S}$, travelling within the solid angle $d\boldsymbol{\Omega}_\nu$ about the flow direction $\boldsymbol{l} = (\boldsymbol{y}-\boldsymbol{x})/|\boldsymbol{y}-\boldsymbol{x}|$ and crossing a unit area, placed around the point $\boldsymbol{y} \in \mathcal{D}$ and normal to $\boldsymbol{l}$.

- The differential form $d\sigma_{\nu\mathcal{D}}$ is defined in such a way that

$$\frac{1}{V_{\mathcal{D}}}\int d\boldsymbol{y}\, d\sigma_{\nu\mathcal{D}} = \sum_{\text{spins}\,\in\,D}\int\frac{d\boldsymbol{y}\, d\mathfrak{P}_d}{2E_\nu} \tag{17}$$

represents the differential cross section of the neutrino scattering off the detector as a whole.

In the particular (and the most basically important) case of neutrino scattering in the reaction $\nu a \to F'_d \ell^-_\beta$, provided that the momentum distribution of the target scatterers a is sufficiently narrow, the differential form $d\sigma_{\nu\mathcal{D}}$ becomes exactly the elementary differential cross section of this reaction multiplied by the total number of the particles a in $\mathcal{D}$.

▷ Now let us address the last sub-integral multiplier of (15b), given by

$$\mathcal{P}_{\alpha\beta}(E_\nu, L) = \sum_{ij} V_{\alpha i}^* V_{\alpha j} V_{\beta i} V_{\beta j}^* S_{ij} \exp\left(i\varphi_{ij} - \mathscr{A}_{ij}^2 - \Theta_{ij}\right), \tag{18}$$

$$\Theta_{ij} = \Theta_i + \Theta_j, \tag{19}$$

$$\Theta_j = \frac{m_j^2}{2\mathfrak{D}^2}\left[(\mathfrak{n}_0 - \mathfrak{n}) + \frac{1}{2}\left(\mathfrak{m} - \mathfrak{n} - \mathfrak{n}^2\right) r_j + \left(\mathfrak{n} + \frac{1}{2}\right)\left(\mathfrak{m} - \mathfrak{n} - \mathfrak{n}^2\right) r_j^2 + \mathcal{O}(r_j^3)\right]. \tag{20}$$

Let's remind that the function $\mathfrak{n}_0$ coincides with $\mathfrak{n}$ in the case of exact energy-momentum conservation in the vertices of our diagram. Therefore in the vicinity of the maximum of the product $\widetilde{\delta}_s(p_\nu - q_s)\widetilde{\delta}_d(p_\nu + q_d)$ (that is at $q_s \approx -q_d \approx p_\nu$), which gives the main contribution into the event rate, one can neglect the alternating quantity $\mathfrak{n}_0 - \mathfrak{n}$ in (20). Taking into account the properties of the function $\mathfrak{n}$ one can also neglect the $\mathcal{O}(r_j^2)$ contributions in (20). In this approximation

$$\Theta_j \approx \frac{m_j^4 R\left(\mathfrak{m} - \mathfrak{n} - \mathfrak{n}^2\right)}{4E_\nu^2} \approx \frac{m_j^4 R\left(\mathfrak{m} - \mathfrak{n}_0 - \mathfrak{n}_0^2\right)}{4E_\nu^2} = \frac{m_j^4\left[R_{00}\mathscr{R} - (\mathbf{Rl})^2\right]}{4RE_\nu^2} \geq 0.$$

• The factor (18) coincides with the QM expression for the neutrino flavor transition probability,

$$\mathcal{P}_{\alpha\beta}^{(\mathrm{QM})}(E_\nu, L) = \sum_{ij} V_{\alpha i} V_{\beta j} V_{\alpha j}^* V_{\beta i}^* \exp\left(i\varphi_{ij}\right). \tag{21}$$

provided that $S_{ij} = 1$, $\Theta_{ij} = 0$, and $\mathscr{A}_{ij} = 0$. So it can be considered as a QFT refinement of the QM result.

BUT!

• A probabilistic interpretation of the function $\mathcal{P}_{\alpha\beta}$ can be only provisionally true, because the factors S_{ij} and $\mathscr{A}_{ij}$ involve the functions $\mathfrak{D}$, $\mathfrak{n}$, and $\mathfrak{m}$ strongly dependent on the neutrino energy E_ν and external momenta $\boldsymbol{p}_\varkappa$; all these (except for the momenta of secondaries in $\mathcal{D}$) are variables of integration in (15b).

As a result, the factor $\mathcal{P}_{\alpha\beta}$, as function of α and β, does not satisfy the unitarity relations

$$\sum_\alpha \mathcal{P}_{\alpha\beta}^{(\mathrm{QM})} = \sum_\beta \mathcal{P}_{\alpha\beta}^{(\mathrm{QM})} = 1,$$

which are a commonplace in the QM theory of neutrino oscillations.

The point is that the domains and shapes of the functions $\mathfrak{D}$, $\mathfrak{n}$, and $\mathfrak{m}$ are essentially different for each of the nine leptonic pairs $(\ell_\alpha, \ell_\beta)$. These differences are governed by kinematics of the subprocesses in $\mathcal{S}$ and $\mathcal{D}$ (in particular, their thresholds), that is, eventually, by the leptonic masses (m_e, m_μ, m_τ) and by the momentum spreads (σ_e, σ_μ, σ_τ) of the leptonic WPs, which are not necessarily equal to each other, perhaps even within an order of magnitude.

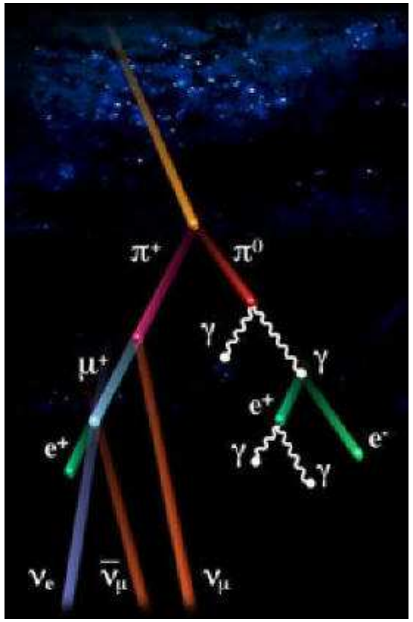
So $\mathcal{P}_{\alpha\beta}(E_\nu, L)$ is not the flavor transition probability!

Having this in mind, we will call it probability factor for short.

Two more drawbacks.

• The probabilistic treatment of $\mathcal{P}_{\alpha\beta}$ is even more problematic in real-life experiments, because the detector event rate (with ℓ_β appearance in our case) is defined by many subprocesses of different types in the source and detector.

E.g., in the astrophysical, atmospheric and accelerator neutrino experiments, the major processes of neutrino production are in-flight decays of light mesons ($\pi_{\mu 2}$, $K_{\mu 2}$, $K_{\mu 3}$, K_{e3}, etc.) and muons, and neutrino interactions with a detector medium consist of an incoherent superposition of exclusive reactions of many types, – from (quasi)elastic to deep-inelastic.

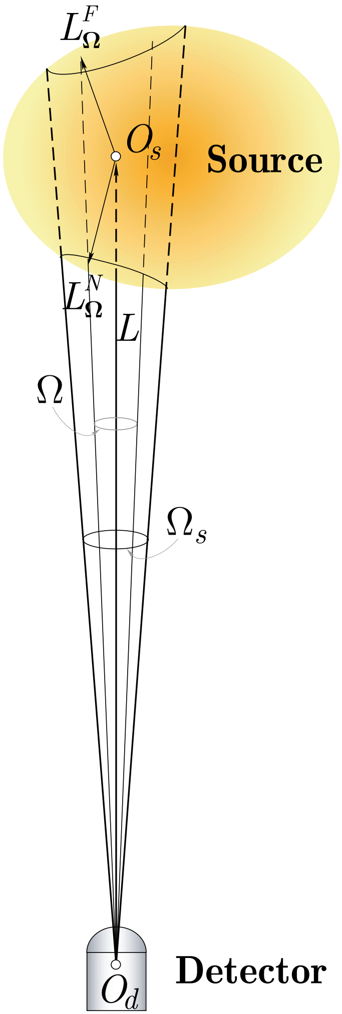


• A "technical" drawback is the dependence of the function S_{ij} (which will be referred to as decoherence factor) on the four "instrumental" time parameters x_1^0, x_2^0, y_1^0, y_2^0.

So far we have made no assumption concerning a "synchronization" of the time windows (x_1^0, x_2^0) and (y_1^0, y_2^0). Thus, it is no wonder that the decoherence factor turns to be vanishingly small in magnitude if these windows are not adjusted to account that the representative time of ultrarelativistic neutrino propagation from $\mathcal{S}$ to $\mathcal{D}$ is equal to the mean distance, $\overline{L}$, between $\mathcal{S}$ and $\mathcal{D}$.

Before discussing the role of the decoherence factor, we perform one more, and the last, simplification of the formula for $dN_{\alpha\beta}$.

Step 3: Spatial averaging.



We'll use again the requirement that the characteristic dimensions of $\mathcal{S}$ and $\mathcal{D}$ are small compared to $\overline{L}$. Under certain conditions, this allows us to replace approximately

$$|\boldsymbol{y} - \boldsymbol{x}| \longmapsto \overline{L} = \frac{1}{2\Omega_s} \int_{\Omega_s} d\boldsymbol{\Omega} \left(L_{\boldsymbol{\Omega}}^{F} + L_{\boldsymbol{\Omega}}^{N} \right),$$

$$d\Phi_\nu \longmapsto d\overline{\Phi}_\nu, \quad d\sigma_{\nu\mathcal{D}} \longmapsto d\overline{\sigma}_{\nu\mathcal{D}}.$$

The range of applicability of this approximation is in general much more limited than that of (15b), as a consequence of additional restrictions implicitly imposed on the distribution functions $\overline{f}_a$, absolute dimensions and geometry of $\mathcal{S}$ and $\mathcal{D}$.
These issues are bit more complicated then the considered above and must be the subject of special attention in the neutrino oscillation experiments.

Finally, we arrive at the very simple but rather rough expression:

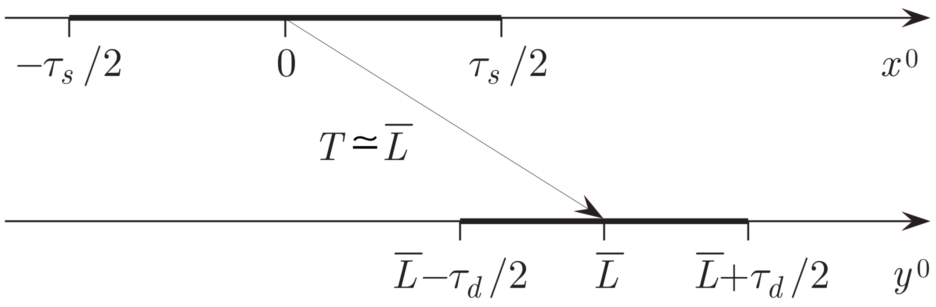
$$dN_{\alpha\beta} = \tau_d \int d\overline{\Phi}_\nu \int d\overline{\sigma}_{\nu\mathcal{D}} \mathcal{P}_{\alpha\beta}(E_\nu, \overline{L}). \tag{22}$$

In particular, it is not applicable to the short base-line experiments.

Synchronized measurements.

Let us now return to the decoherence factor, limiting ourselves to a consideration of "synchronized" measurements, in which

$$x^0_{1,2} = \mp\frac{\tau_s}{2}, \quad y^0_{1,2} = \overline{L} \mp \frac{\tau_d}{2}.$$



With certain technical simplifications, the factor (13) can be expressed through a real-valued function $S(t,t',b)$ of three dimensionless variables, namely:

$$S_{ij} = S\left(\mathfrak{D}\tau_s, \mathfrak{D}\tau_d, \mathscr{B}_{ij}\right),$$

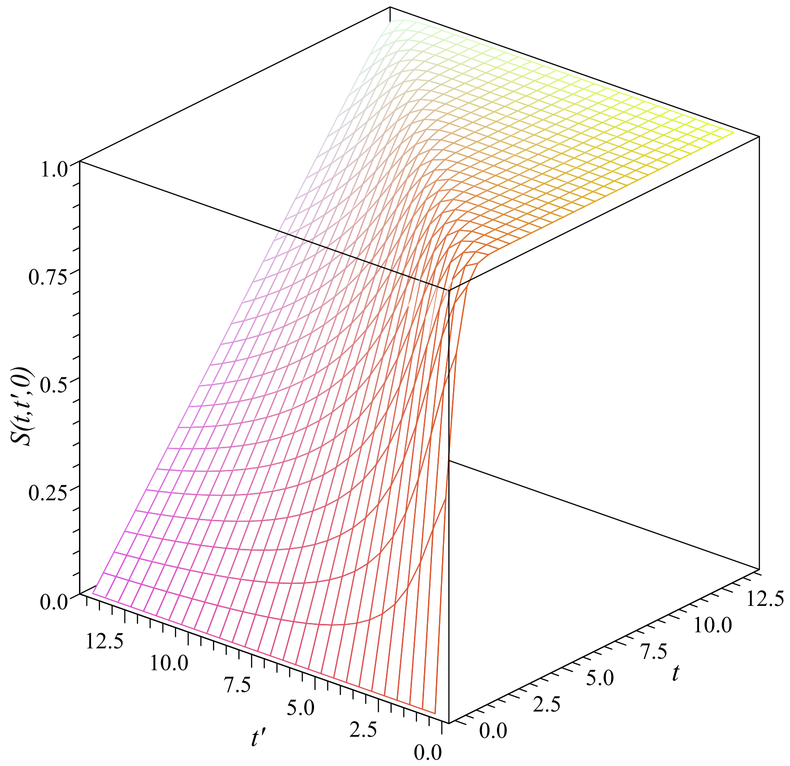
$$2t'S(t,t',b) = \exp\left(-b^2\right)\,\mathrm{Re}\left[\mathsf{lerf}\left(t+t'+ib\right) - \mathsf{lerf}\left(t-t'+ib\right)\right].$$

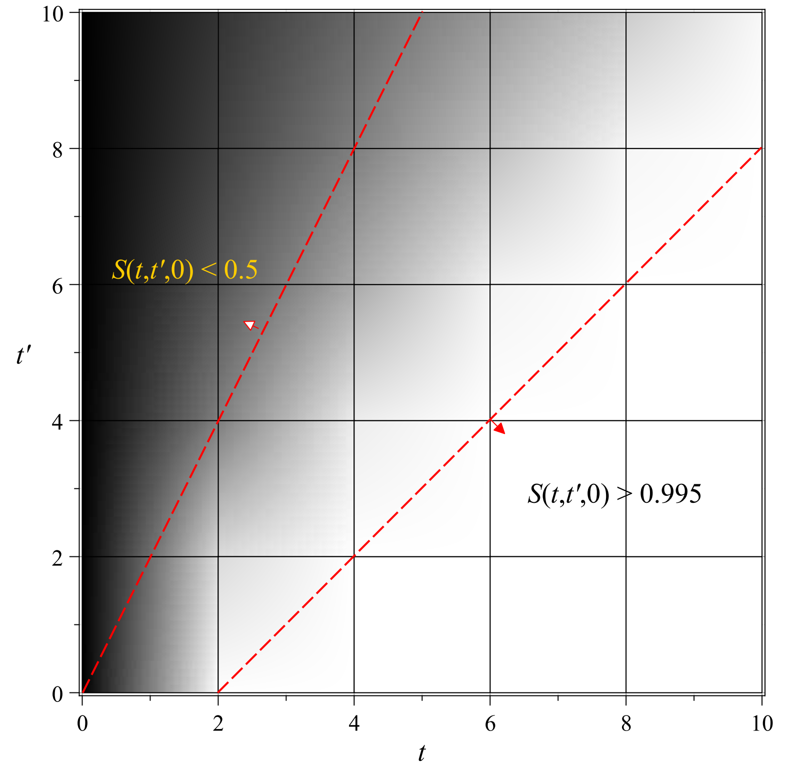
Diagonal decoherence function.

$$S(t,t',0) = \frac{1}{2t'}\left[\mathsf{lerf}\left(t+t'\right) - \mathsf{lerf}\left(t-t'\right)\right] \equiv S_0(t,t'), \tag{23}$$

This function corresponds to the noninterference (neutrino mass independent) decoherence factors S_{ii}. The following inequalities can be proved:

$$0 < S_0(t,t') < 1, \quad S_0(t,t') < t/t' \text{ for } t' \geq t, \quad S_0(t+\delta t, t) > \mathsf{erf}(\delta t) \text{ for } \delta t > 0.$$



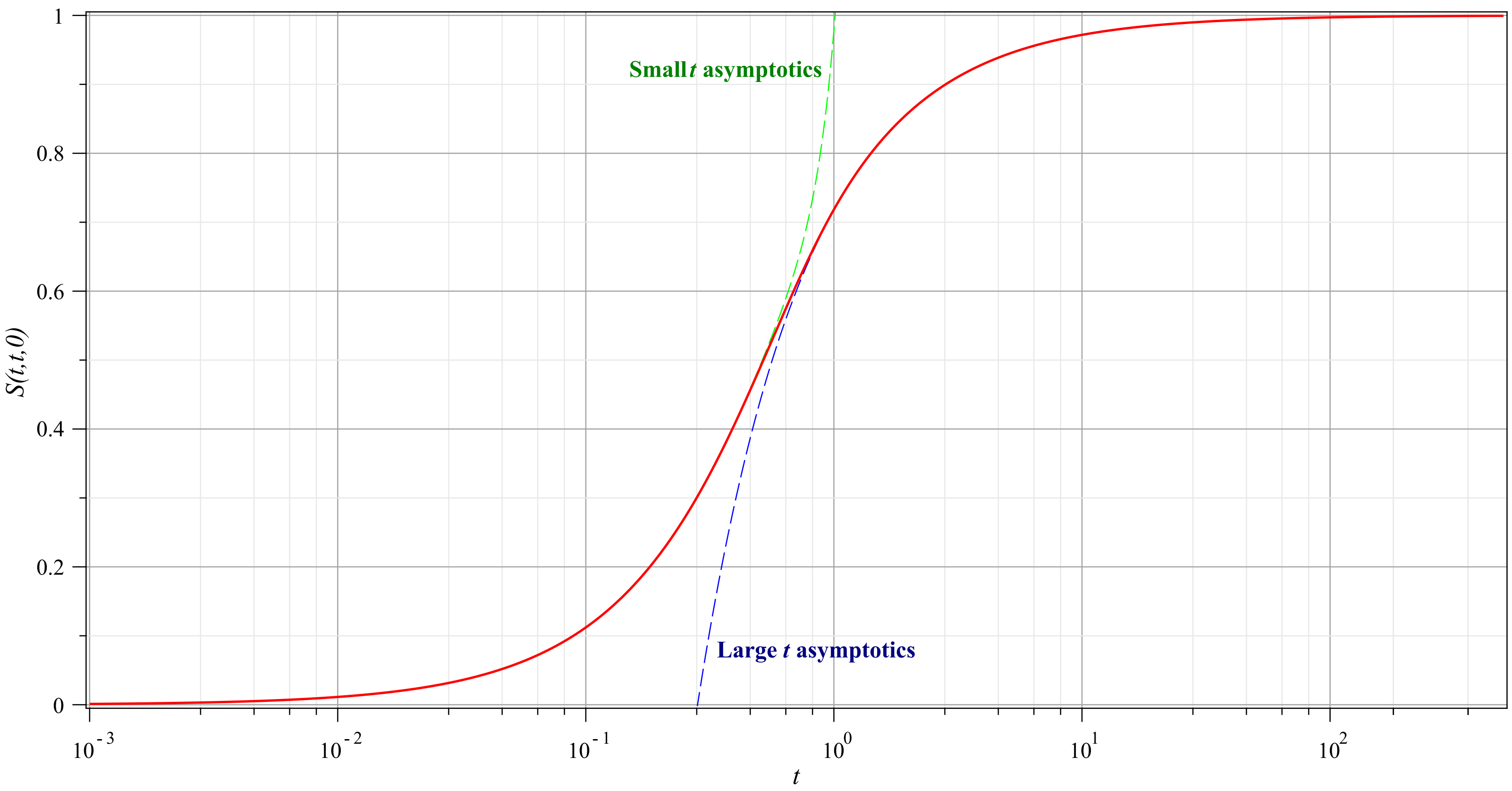


The strong dependence of the common suppression factor $S_0(t,t')$ on its arguments at $t \lesssim t'$ provides a potential possibility of an experimental estimation of the function $\mathfrak{D}$ (or, rather, of its mean values within the phase spaces), based on the measuring the count rate $dR_{\alpha\beta} = dN_{\alpha\beta}/\tau_d$ as a function of τ_d and τ_s (at fixed $\overline{L}$) and comparing the data with the results of Monte-Carlo simulations.

The optimal strategy of such an experiment should be a subject of a dedicated analysis.

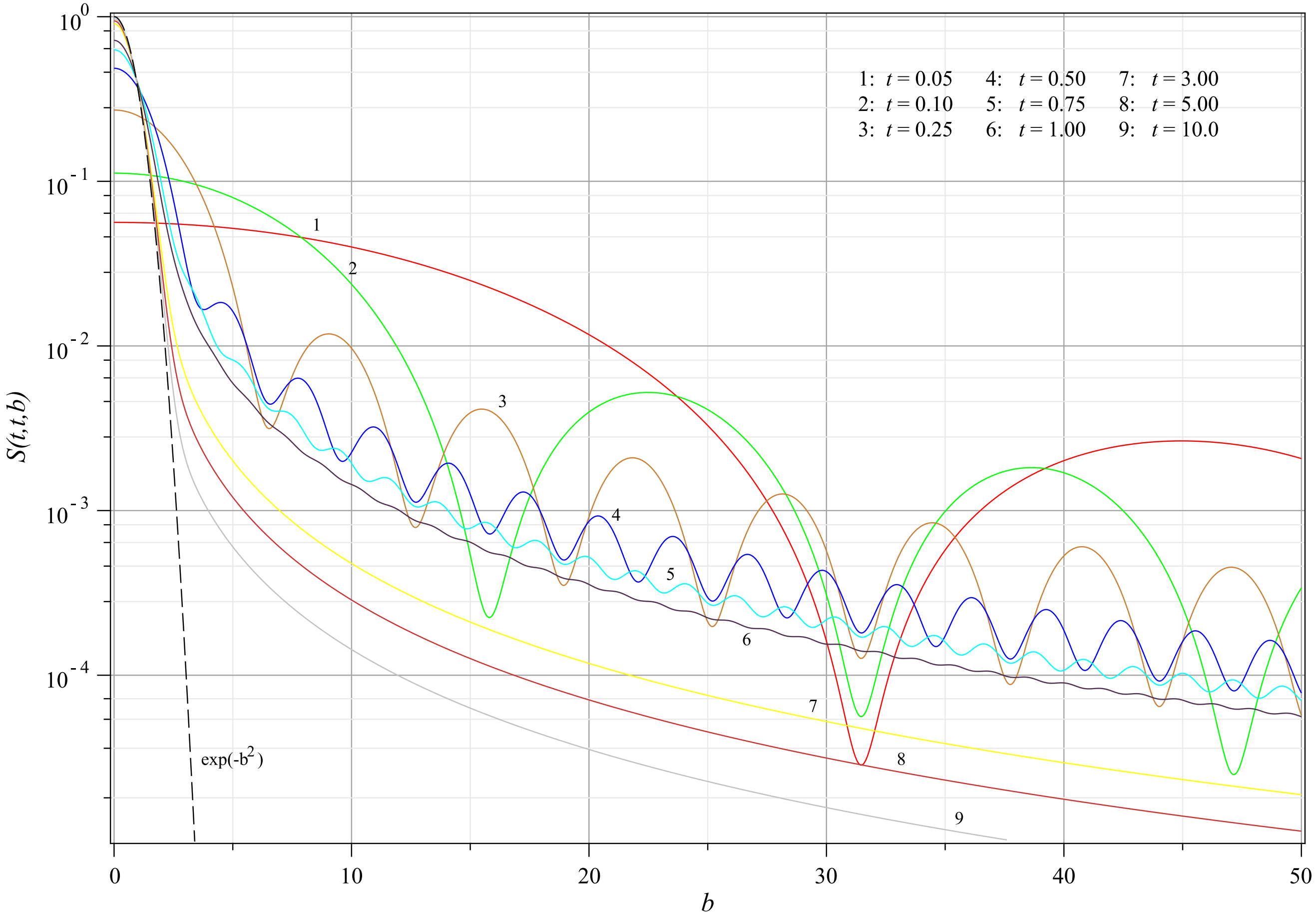
For the important special case, $t' = t$ (representative, in particular, for the experiments with accelerator neutrino beams), we find

$$S_0(t,t) = \mathsf{erf}(2t) - \frac{1-e^{-4t^2}}{2\sqrt{\pi}t} \approx \begin{cases} \dfrac{2t}{\sqrt{\pi}}\left(1 - \dfrac{2t^2}{3} + \dfrac{8t^4}{15}\right) & \text{for } \ t \ll 1, \\ 1 - \dfrac{1}{2\sqrt{\pi}t} & \text{for } \ t \gg 1. \end{cases} \tag{24}$$

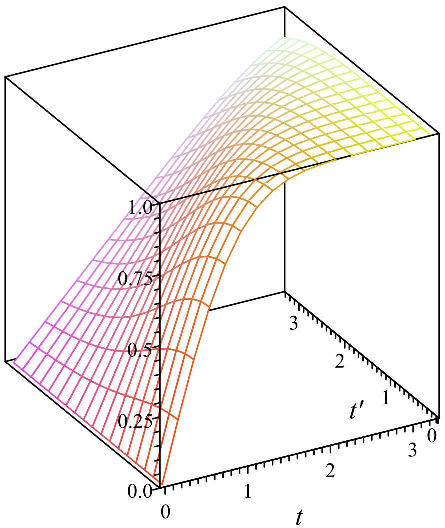


Nondiagonal decoherence function.

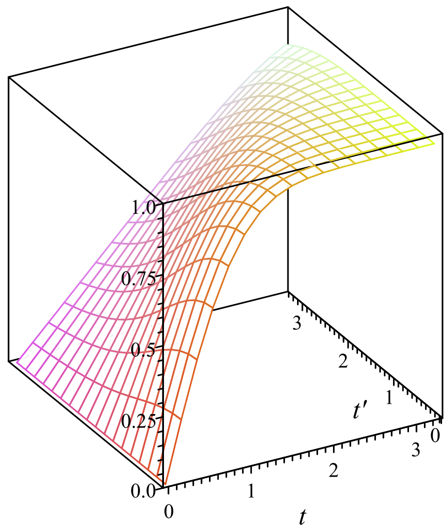
The decoherence function $S(t, t', b)$ at $b \neq 0$ is much more involved.



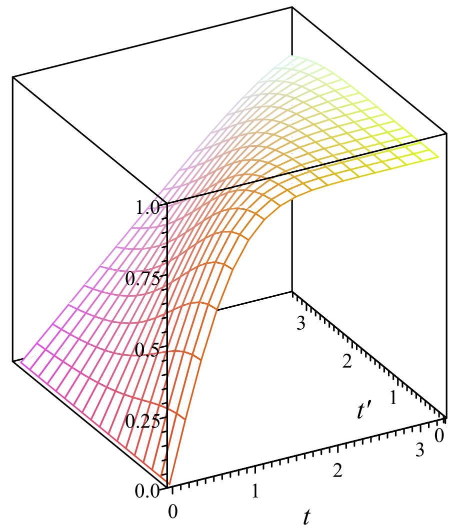
At very large t, the function $S(t, t, b)$ becomes nearly independent on t, slowly approaching the asymptotic behavior $S(t, t, b) \sim \exp(-b^2)$ $(t, t' \to \infty)$.



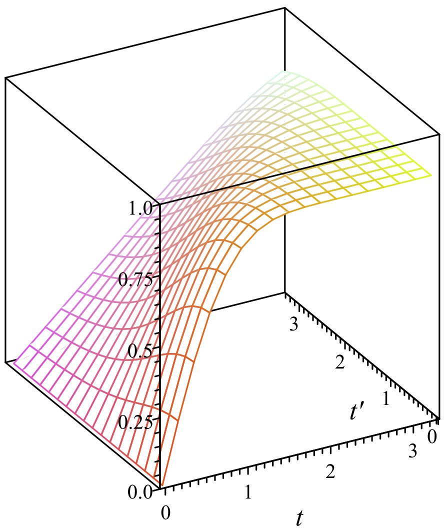
$S(t, t', 0.1)$.



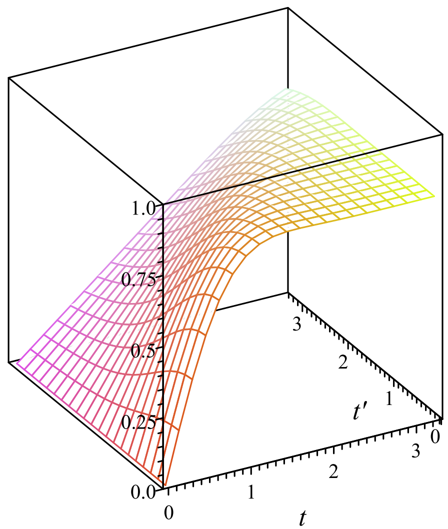
$S(t, t', 0.2)$.



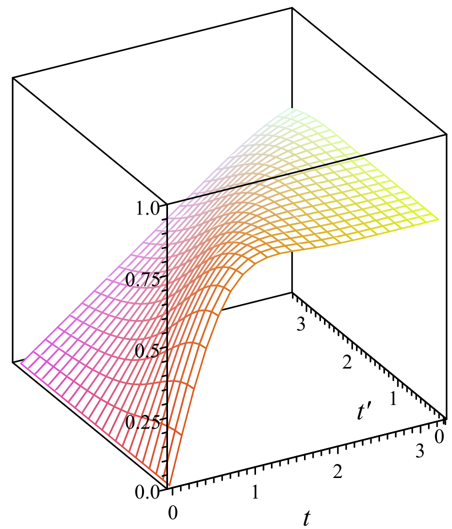
$S(t, t', 0.3)$.



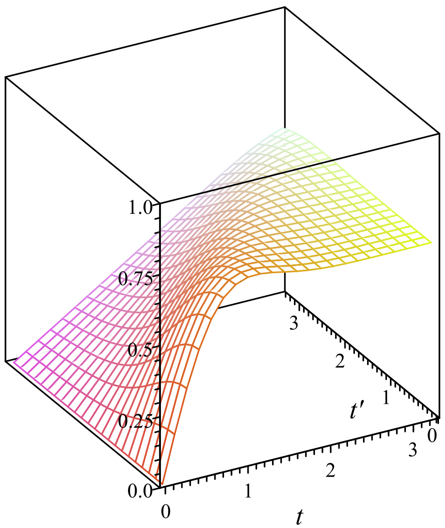
$S(t, t', 0.4)$.



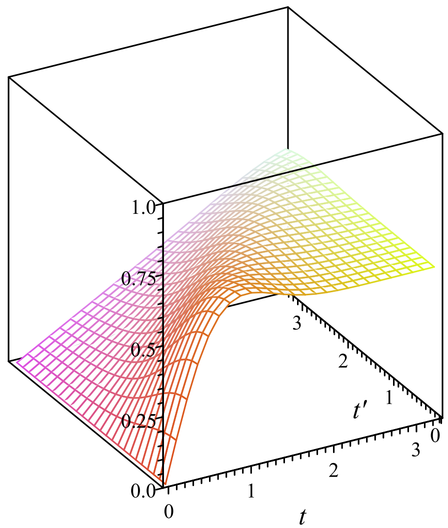
$S(t, t', 0.5)$.



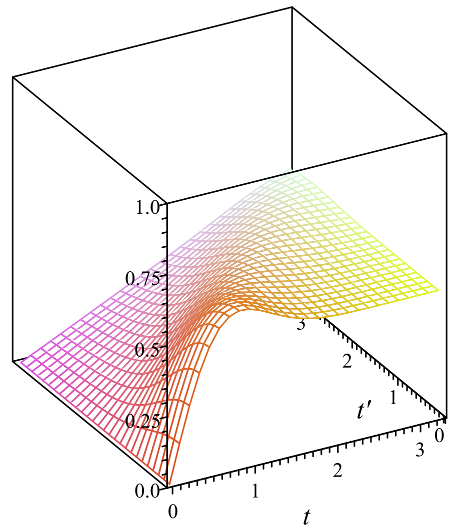
$S(t, t', 0.6)$.



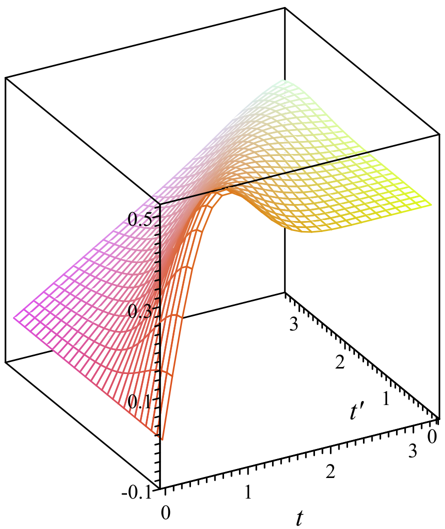
$S(t, t', 0.7)$.



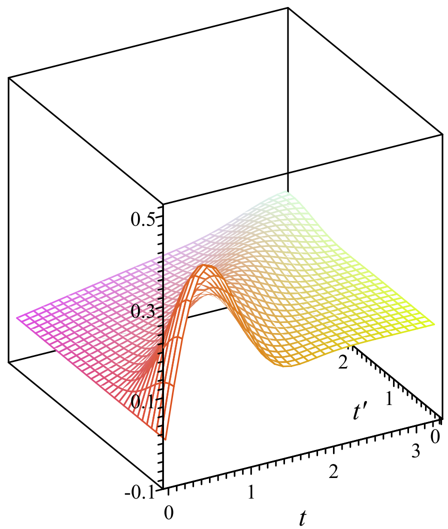
$S(t, t', 0.8)$.



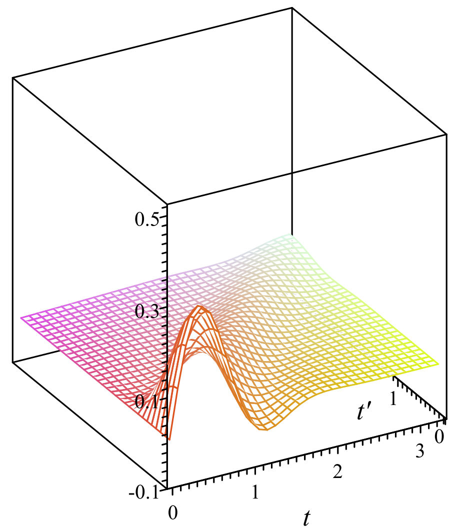
$S(t, t', 0.9)$.



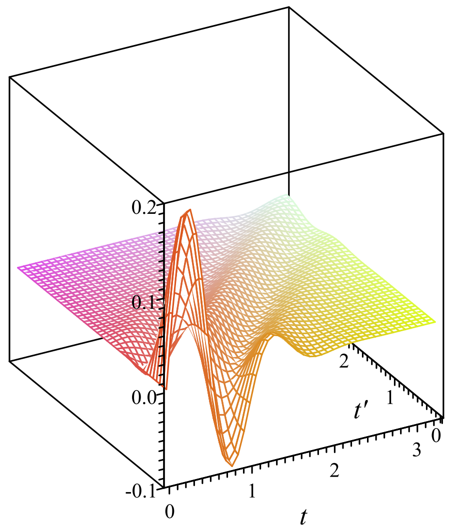
$S(t, t', 1.0)$.



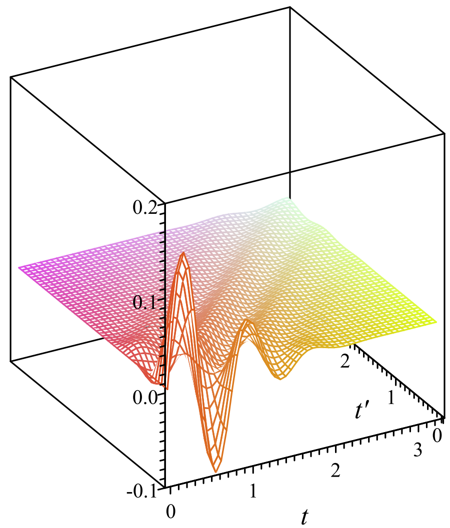
$S(t, t', 1.5)$.



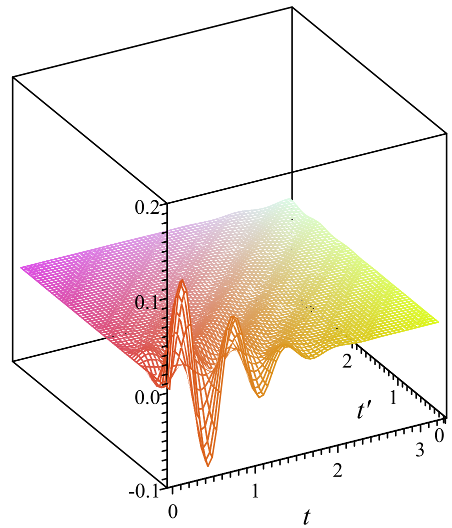
$S(t, t', 2.0)$.



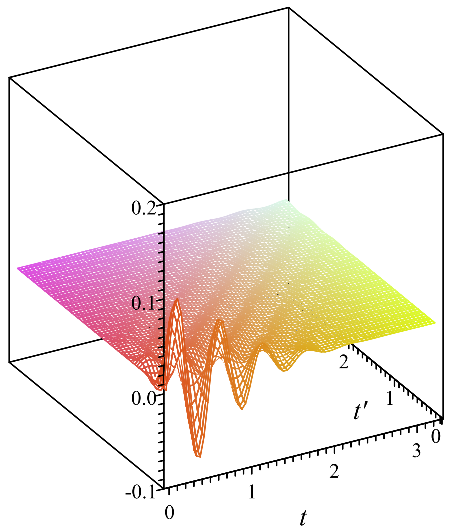
$S(t, t', 3.0)$.



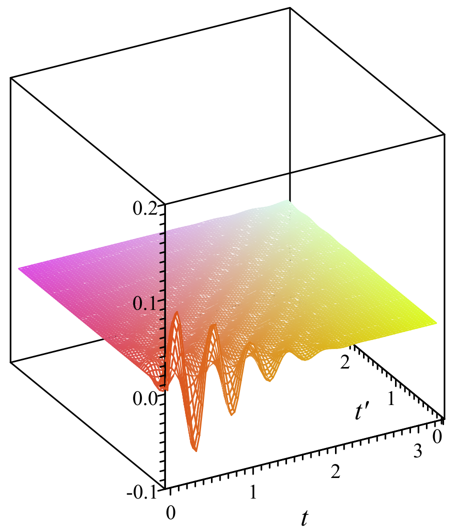
$S(t, t', 4.0)$.



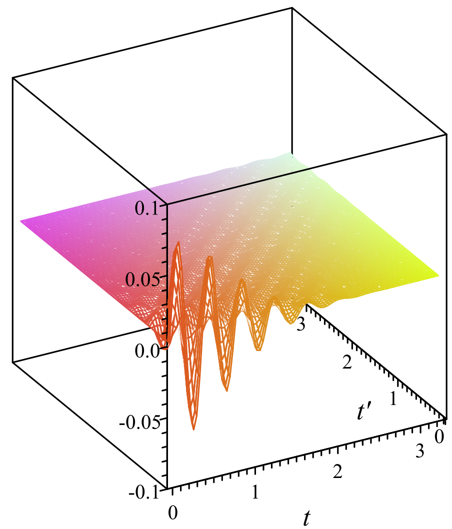
$S(t, t', 5.0)$.



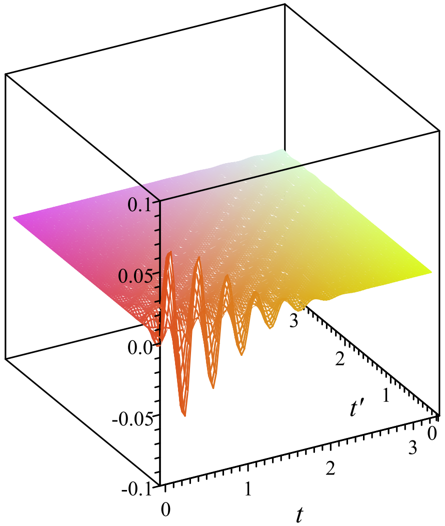
$S(t, t', 6.0)$.



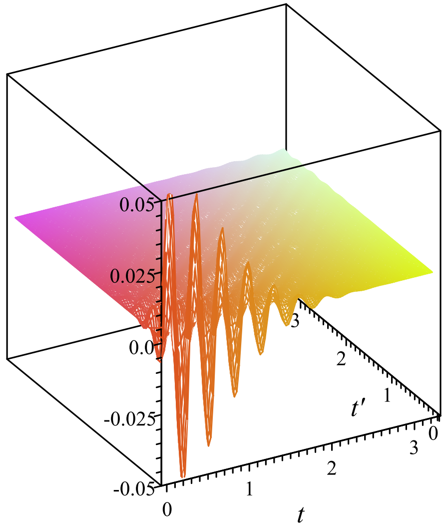
$S(t, t', 7.0)$.



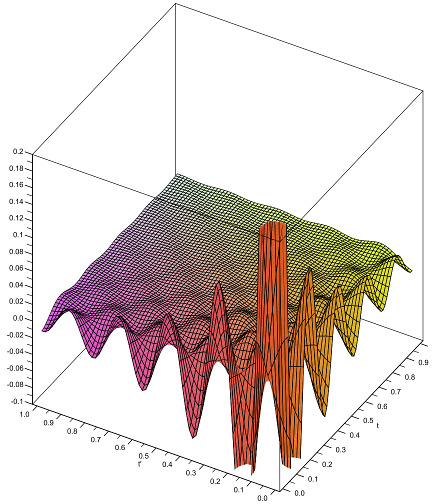
$S(t, t', 8.0)$.

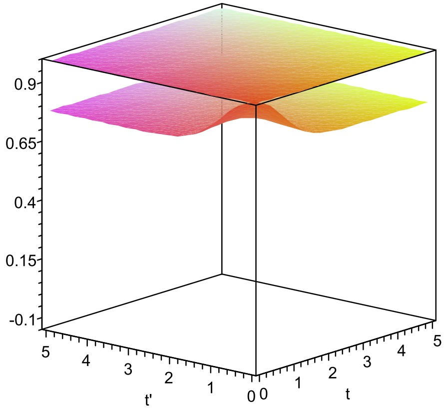


$S(t, t', 9.0)$.

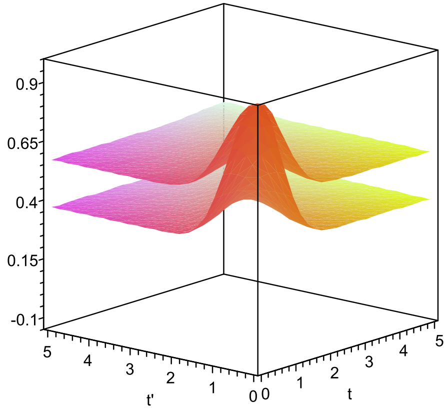


$S(t, t', 10.0)$.

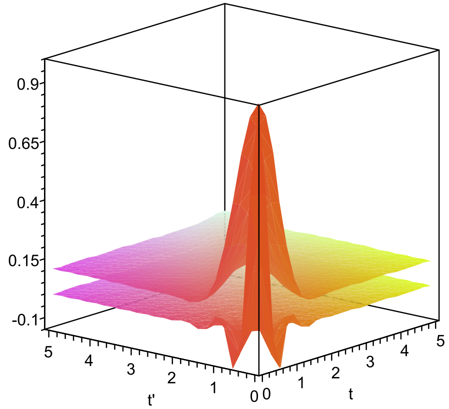

$S(t, t', 15.0)/S_0(t, t')$.



$S(t, t', 0.10)/S_0(t, t')$,
$S(t, t', 0.50)/S_0(t, t')$.



$S(t, t', 0.75)/S_0(t, t')$,
$S(t, t', 1.00)/S_0(t, t')$.



$S(t, t', 1.50)/S_0(t, t')$,
$S(t, t', 4.00)/S_0(t, t')$.

Flavor transitions in the asymptotic regime.

In the asymptotic regime,

$$S(t,t',b) \sim \exp(-b^2) \quad (t,t' \to \infty).$$

the probability factor (18) takes on the form already known from the literature,[a]

$$\boxed{\mathcal{P}_{\alpha\beta}(E_\nu,\overline{L}) = \sum_{ij} V_{\alpha i}^* V_{\alpha j} V_{\beta i} V_{\beta j}^* \exp\left(i\varphi_{ij} - \mathscr{A}_{ij}^2 - \mathscr{B}_{ij}^2 - \Theta_{ij}\right),} \tag{25}$$

but with the essential difference that the factors $\mathscr{A}_{ij}$, $\mathscr{B}_{ij}$ and Θ_{ij} do depend (through the functions $\mathfrak{D}$, $\mathfrak{n}$, and $\mathfrak{m}$) on the neutrino energy and momenta of the external WPs.

This dependence drastically affects the magnitude and shape of these factors if at least some of the WPs have relativistic momenta (that is always the case in the contemporary neutrino oscillation experiments). For sufficiently small and/or hierarchically different momentum spreads $\sigma_\varkappa$, the functions $\mathscr{A}_{ij}$ and $\mathscr{B}_{ij}$ may vary in many orders of magnitude through their multidimensional domain.

[a] See, e.g., C. Giunti C and C. W. Kim, *Fundamentals of Neutrino Physics and Astrophysics* (Oxford University Press Inc., New York, 2007); M. Beuthe, *Oscillations of neutrinos and mesons in quantum field theory*, Phys. Rept. **375** (2003) 105 (arXiv:hep-ph/0109119); M. Beuthe, *Towards a unique formula for neutrino oscillations in vacuum*, Phys. Rev. D **66** (2002) 013003 (arXiv:hep-ph/0202068).

Major properties of the transition "probability".

- The factors $\exp\left(-\mathscr{A}_{ij}^2\right)$ (with $i\neq j$) suppress the interference terms at the distances exceeding the "coherence length"

$$L_{ij}^{\text{coh}} = \frac{1}{\Delta v_{ij}\mathfrak{D}} \gg |L_{ij}| \qquad (\Delta v_{ij} = |v_j - v_i|),$$

when the νWPs $\psi_{X_d}^i(\mathbf{p}_i, X_s - X_d)$ and $\psi_{X_d}^j(\mathbf{p}_j, X_s - X_d)$ are strongly separated in space and do not interfere anymore. Clearly $L_{ij}^{\text{coh}} \to \infty$ in the plane-wave limit.

- The suppression factors $\exp\left(-\mathscr{B}_{ij}^2\right)$ $(i\neq j)$ work in the opposite situation, when the external packets in $\mathcal{S}$ or $\mathcal{D}$ (or in both $\mathcal{S}$ and $\mathcal{D}$) are strongly delocalized

The gross dimension of the the neutrino production and absorption regions in $\mathcal{S}$ and $\mathcal{D}$ is of the order of $1/\mathfrak{D}$. The interference terms vanish if this scale is large compared to the "interference length"

$$L_{ij}^{\text{int}} = \frac{1}{4\Delta E_{ij}} = \frac{2L_{ij}}{\pi\mathfrak{n}}.$$

In other words, the QFT approach predicts vanishing of neutrino oscillations in the plane-wave limit. In this limit, the flavor transition probability does not depend on $\overline{L}$, E_ν, and neutrino masses m_i and becomes

$$\mathcal{P}_{\alpha\beta}^{\mathsf{PWL}} = \sum_i |V_{\alpha i}|^2 |V_{\beta i}|^2 \leq 1.$$

Thereby, a nontrivial interference of the diagrams with the intermediate neutrinos of different masses is only possible if $\mathfrak{D} \neq 0$.

- Our detailed analysis of the generic subprocesses $1 \to 2$, $1 \to 3$, and $2 \to 2$ shows that $\mathfrak{D} \neq 0$ if in both vertices of the macrodiagram there are at least two interacting WPs $\varkappa$ (no matter in or out) with $\sigma_\varkappa \neq 0$.

- The same requirement unavoidably leads to the vanishing of the non-diagonal terms, when the mean distance between $\mathcal{S}$ and $\mathcal{D}$ becomes large enough in comparison with the coherence lengths L_{ij}^{coh}.

- As a result, the range of applicability of the standard QM formula for the neutrino oscillations probability is limited by rather restrictive conditions,

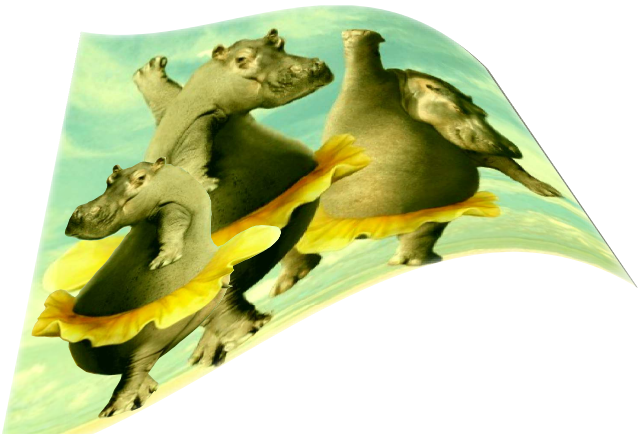
$$\left\langle \left(\frac{2\pi\mathfrak{D}L}{E_\nu L_{ij}} \right)^2 \right\rangle \ll 1, \quad \left\langle \left(\frac{\pi\mathfrak{n}}{2\mathfrak{D}L_{ij}} \right)^2 \right\rangle \ll 1, \quad \text{and} \quad \langle |\Theta_{ij}| \rangle \ll 1.$$

The angle brackets symbolize an averaging over the phase subspace of the process (3) which provides the main contribution into the measured count rate.

The obtained conditions were obtained under a number of assumptions and simplifications, which are not necessarily adequate to fully represent the real-life experimental conditions. Our consideration suggests that in the analysis and interpretation of real data one should take into account the operating times of the source and detector, their geometry and dimensions, explicit form of the distribution functions of in-packets, and other technical details.

Intermediary conclusions on the QFT approach.

- The standard QM ν-oscillation formula has rather limited range of applicability.
- The QFT modifications drastically depend upon:



  - ▷ momentum spreads of the external “in” and “out” wave packets (determined by the environment and “prehistory” of their creation).
  - ▷ reaction types in the neutrino production and absorption regions [“source” and ”detector”, respectively] and phase-space domains of these reactions;
  - ▷ time interval of steady-state operation of the source “machine ” and detector exposure time;
  - ▷ dimensions of the source and detector and distance between them.

- Essentially all QFT effects are decoherent and thus lead to a “smoothing”, distortion or vanishing of the interference (oscillating) terms and to a general suppression of the neutrino event rate in the detector. This suppression is potentially measurable in the dedicate experiments.

The predicted effects are usually small. But “small” does not mean “uninteresting”.

Modern physics flourishes due mainly to discovering very small effects.