SEE-SAW
An Elementary Introduction

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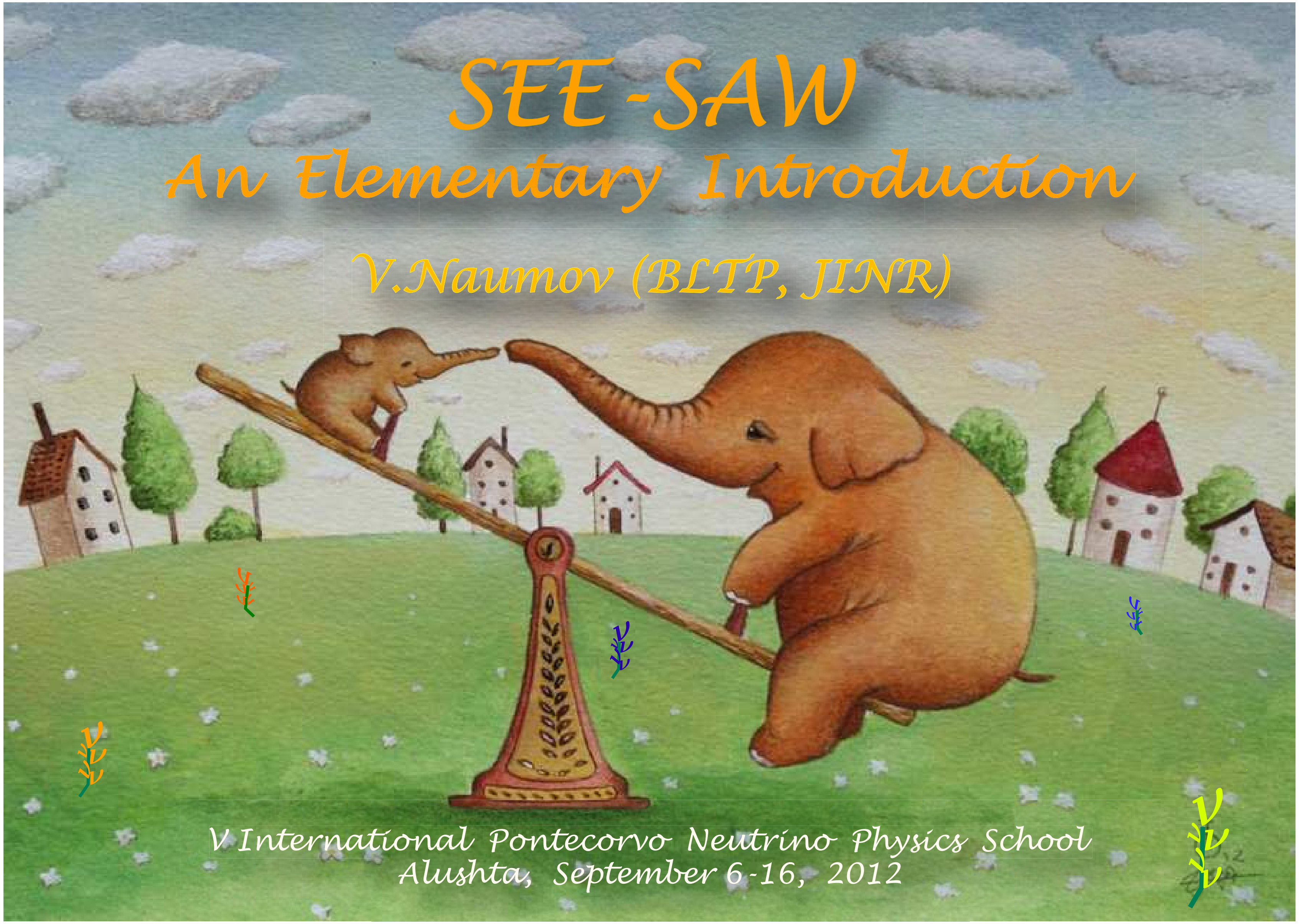

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For more details, see

814 papers with key "`f t seesaw or see-saw`" and
4043 papers with key "`f t neutrino mass* not seesaw not see-saw`"

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and references therein.

Interaction Lagrangian and weak currents

In the Standard Model (SM), the charged and neutral current neutrino interactions are described by the following parts of the full Lagrangian:

$$\mathcal{L}_I^{\mathsf{CC}}(x) = -\frac{g}{2\sqrt{2}} j_\alpha^{\mathsf{CC}}(x) W^\alpha(x) + \mathsf{H.c.} \quad \text{and} \quad \mathcal{L}_I^{\mathsf{NC}}(x) = -\frac{g}{2\cos\theta_{\mathsf{W}}} j_\alpha^{\mathsf{NC}}(x) Z^\alpha(x).$$

Here g is the $SU(2)$ (electro-weak) gauge coupling constant

$$g^2 = 4\sqrt{2} m_W^2 G_F, \quad g \sin\theta_{\mathsf{W}} = |e|$$

and θ_{W} is the weak mixing (Weinberg) angle ($\sin^2\theta_{\mathsf{W}}(M_Z) = 0.23120$).
The leptonic charged current and neutrino neutral current are given by the expressions:

$$j_\alpha^{\mathsf{CC}}(x) = 2 \sum_{\ell=e,\mu,\tau,\ldots} \overline{\nu}_{\ell,L}(x) \gamma_\alpha \ell_L(x) \quad \text{and} \quad j_\alpha^{\mathsf{NC}}(x) = \sum_{\ell=e,\mu,\tau,\ldots} \overline{\nu}_{\ell,L}(x) \gamma_\alpha \nu_{\ell,L}(x).$$

The currents may include (yet unknown) heavy neutrinos and corresponding charged leptons. The left- and right-handed fermion fields are defined as usually:

$$\nu_{\ell,L/R}(x) = \left(\frac{1 \pm \gamma_5}{2}\right) \nu_\ell(x) \quad \text{and} \quad \ell_{L/R}(x) = \left(\frac{1 \pm \gamma_5}{2}\right) \ell(x).$$

Note that the kinetic term of the Lagrangian includes both L and R handed neutrinos and moreover, it can include other sterile neutrinos:

$$\begin{aligned}\mathcal{L}_0 =& \frac{i}{2}\left[\overline{\nu}(x)\gamma^\alpha\partial_\alpha\nu(x) - \partial_\alpha\overline{\nu}(x)\gamma^\alpha\nu(x)\right] \equiv \frac{i}{2}\overline{\nu}(x)\overleftrightarrow{\boldsymbol{\partial}}\nu(x)\\ =& \frac{i}{2}\left[\overline{\nu}_L(x)\overleftrightarrow{\boldsymbol{\partial}}\nu_L(x) + \overline{\nu}_R(x)\overleftrightarrow{\boldsymbol{\partial}}\nu_R(x)\right],\end{aligned}$$

$$\boldsymbol{\nu}(x) = \boldsymbol{\nu}_L(x) + \boldsymbol{\nu}_R(x) = \begin{pmatrix}\nu_e(x)\\ \nu_\mu(x)\\ \nu_\tau(x)\\ .\\ .\\ .\end{pmatrix}, \quad \boldsymbol{\nu}_{L/R}(x) = \begin{pmatrix}\nu_{e,L/R}(x)\\ \nu_{\mu,L/R}(x)\\ \nu_{\tau,L/R}(x)\\ .\\ .\\ .\end{pmatrix} = \frac{1\pm\gamma_5}{2}\begin{pmatrix}\nu_e(x)\\ \nu_\mu(x)\\ \nu_\tau(x)\\ .\\ .\\ .\end{pmatrix}.$$

Neutrino chirality: $\gamma_5\nu_L = -\nu_L$ and $\gamma_5\nu_R = +\nu_R$.

The Lagrangian of the theory with massless neutrinos is invariant with respect to the global gauge transformations

$$\nu_\ell(x) \to e^{i\Lambda_\ell}\nu_\ell(x), \quad \ell(x) \to e^{i\Lambda_\ell}\ell(x) \quad \text{with} \quad \Lambda_\ell = \text{const}.$$

This leads (through 1st Noether's theorem) to conservation of the individual lepton flavor numbers L_ℓ (electron, muon, tauon, etc.). It is not the case for massive neutrinos.

There are two types of possible neutrino mass terms: Dirac and Majorana.

Dirac neutrinos

The conventional Dirac mass term for a single spinor field $\psi(x)$ is well known:

$$-m\overline{\psi}(x)\psi(x) = -m\left[\overline{\psi}_R(x)\psi_L(x) + \overline{\psi}_L(x)\psi_R(x)\right] = -m\overline{\psi}_R(x)\psi_L(x) + \text{H.c.}$$

The most general extension of this construction to the N-generation Dirac neutrino case reads:

$$\mathcal{L}_{\text{D}}(x) = -\overline{\boldsymbol{\nu}}_R(x)\mathbf{M}_{\text{D}}\boldsymbol{\nu}_L(x) + \text{H.c.},$$

where $\mathbf{M}_{\text{D}}$ is a nonsingular [to exclude massless neutrinos] complex $N \times N$ matrix.

> In general, $N \geq 3$ since the column $\boldsymbol{\nu}_L$ may include both *active* and *sterile* neutrino fields which do not enter into the standard charged and neutral currents.

Any nonsingular complex matrix can be diagonalized by means of an appropriate *bi-unitary* transformation

$$\mathbf{M}_{\text{D}} = \widetilde{\mathbf{V}}\mathbf{m}\mathbf{V}^\dagger, \quad \mathbf{m} = ||m_k\delta_{kl}|| = \text{diag}\,(m_1, m_2, \ldots, m_N),$$

where $\mathbf{V}$ and $\widetilde{\mathbf{V}}$ are unitary matrices and $m_k \geq 0$. Therefore

$$\mathcal{L}_{\text{D}}(x) = -\overline{\boldsymbol{\nu}'}_R(x)\mathbf{m}\boldsymbol{\nu}'_L(x) + \text{H.c.} = -\overline{\boldsymbol{\nu}'}(x)\mathbf{m}\boldsymbol{\nu}'(x) = -\sum_{k=1}^{N} m_k\overline{\nu}_k(x)\nu_k(x),$$

where the new fields ν_k are defined by

$$\boldsymbol{\nu}'_L(x) = \mathbf{V}^\dagger\boldsymbol{\nu}_L(x), \quad \boldsymbol{\nu}'_R(x) = \widetilde{\mathbf{V}}^\dagger\boldsymbol{\nu}_R(x), \quad \boldsymbol{\nu}'(x) = (\nu_1, \nu_2, \ldots, \nu_N)^T.$$

The fields $\boldsymbol{\nu}'_R(x)$ do not enter into $\mathcal{L}_I \Longrightarrow$ the matrix $\widetilde{\mathbf{V}}$ remains out of play...

Since

$$\mathbf{V}\mathbf{V}^\dagger = \mathbf{V}^\dagger\mathbf{V} = \mathbf{1} \quad \text{and} \quad \widetilde{\mathbf{V}}^\dagger\widetilde{\mathbf{V}} = \widetilde{\mathbf{V}}^\dagger\widetilde{\mathbf{V}} = \mathbf{1}$$

the neutrino kinetic term in the Lagrangian is transformed to

$$\mathcal{L}_0 = \frac{i}{2}\left[\overline{\nu}'_L(x)\overleftrightarrow{\boldsymbol{\partial}}\nu'_L(x) + \overline{\nu}'_R(x)\overleftrightarrow{\boldsymbol{\partial}}\nu'_R(x)\right] = \frac{i}{2}\overline{\boldsymbol{\nu}'}(x)\overleftrightarrow{\boldsymbol{\partial}}\boldsymbol{\nu}'(x) = \frac{i}{2}\sum_k \overline{\nu}_k(x)\overleftrightarrow{\boldsymbol{\partial}}\nu_k(x).$$

$$\Downarrow$$

$\nu_k(x)$ is the field of a Dirac neutrino with the mass m_k and the flavor LH neutrino fields $\nu_{\ell,L}(x)$ involved into the SM weak lepton currents are linear combinations of the LH components of the fields of the neutrinos with definite masses:

$$\boxed{\boldsymbol{\nu}_L = \mathbf{V}\boldsymbol{\nu}'_L \quad \text{or} \quad \nu_{\ell,L} = \sum_k V_{\ell k}\nu_{k,L}.}$$

The matrix $\mathbf{V}$ is referred to as the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) neutrino mixing matrix while the matrix $\widetilde{\mathbf{V}}$ is not honored with a personal name.

Quark-lepton complementarity (QLC): Of course the PMNS matrix it is not the same as the CKM (Cabibbo-KobayashiMaskawa) quark mixing matrix. However the PMNS and CKM matrices may be, in a sense, *complementary* to each other.

The QLC means that in the same (PDG) parametrizations the (small) quark and (large) lepton mixing angles satisfy the *empirical* (perhaps accidental) relations:

$$\theta_{12}^{\text{CKM}} + \theta_{12}^{\text{PMNS}} \simeq \pi/4, \quad \theta_{23}^{\text{CKM}} + \theta_{23}^{\text{PMNS}} \simeq \pi/4.$$

Parametrization of mixing matrix for Dirac neutrinos

It is well known that a complex $n \times n$ unitary matrix depends on n^2 *real* parameters.

The classical result by Francis Murnaghan [F. D. Murnaghan, "The unitary and rotation groups (Lectures on Applied Mathematics, Volume 3)," Spartan Books, Washington, D.C. (1962)] states that any $n \times n$ matrix from the unitary group $U(n)$ can be presented as product of the diagonal phase matrix

$$\boldsymbol{\Gamma} = \mathrm{diag}\left(e^{i\alpha_1}, e^{i\alpha_2}, \ldots, e^{i\alpha_n}\right),$$

containing n phases α_k, and $n(n-1)/2$ matrices $\mathbf{U}$ whose main building blocks have the form

$$\begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ -\sin\theta\, e^{+i\phi} & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e^{+i\phi} \end{pmatrix} \underbrace{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}}_{\text{Euler rotation}} \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\phi} \end{pmatrix}.$$

Therefore any $n \times n$ unitary matrix can be parametrized in terms of

$n(n-1)/2$ "angles" (taking values within $[0, \pi/2]$)

and

$n(n+1)/2$ "phases" (taking values within $[0, 2\pi)$).

The usual parametrization of both the CKM and PMNS matrices is of this type.

IMPORTANT: Murnaghan's factorization method does not specify the sequence of the building blocks $\boldsymbol{\Gamma}$ and $\mathbf{U}$.

One can reduce the number of the phases further by taking into account that the Lagrangian with the Dirac mass term is invariant with respect to the transformation

$$\ell \mapsto e^{ia_\ell}\ell, \quad \nu_k \mapsto e^{ib_k}\nu_k, \quad V_{\ell k} \mapsto e^{i(b_k - a_\ell)}V_{\ell k},$$

and to the global gauge transformation

$$\ell \mapsto e^{i\Lambda}\ell, \quad \nu_k \mapsto e^{i\Lambda}\nu_k, \quad \text{with} \quad \Lambda = \text{const}. \tag{1}$$

Therefore $2N - 1$ phases are unphysical and the number of physical (Dirac) phases is

$$n_\mathsf{D} = \frac{N(N+1)}{2} - (2N-1) = \frac{N^2 - 3N + 2}{2} = \frac{(N-1)(N-2)}{2} \qquad (N \geq 2);$$
$$n_\mathsf{D}(2) = 0, \quad n_\mathsf{D}(3) = 1, \quad n_\mathsf{D}(4) = 3, \ldots$$

- The global symmetry (1) leads to conservation of the lepton charge

$$L = \sum_{\ell = e, \mu, \tau, \ldots} L_\ell$$

common to all charged leptons and all neutrinos ν_k. However

$\boxed{\text{The individual lepton flavor numbers } L_\ell \text{ are no longer conserved.}}$

- The nonzero physical phases lead to the CP and T violation in the neutrino sector.

Three-neutrino case

In the most interesting (today!) case of three lepton generations one defines the orthogonal rotation matrices in the ij-planes which depend upon the mixing angles θ_{ij}:

$$\mathbf{O}_{12} = \underbrace{\begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{Solar matrix}}, \quad \mathbf{O}_{13} = \underbrace{\begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{pmatrix}}_{\text{Reactor matrix}}, \quad \mathbf{O}_{23} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}}_{\text{Atmospheric matrix}},$$

(where $c_{ij} \equiv \cos\theta_{ij}$, $s_{ij} \equiv \sin\theta_{ij}$) and the diagonal matrix with the Dirac phase factor:

$$\boldsymbol{\Gamma}_{\mathsf{D}} = \mathsf{diag}\left(1, 1, e^{i\delta}\right).$$

The parameter δ is commonly referred to as the Dirac CP-violation/violating phase.

Finally, by applying Murnaghan's factorization, the PMNS matrix for the Dirac neutrinos can be parametrized as

$$\mathbf{V}_{(\mathsf{D})} = \mathbf{O}_{23}\boldsymbol{\Gamma}_{\mathsf{D}}\mathbf{O}_{13}\boldsymbol{\Gamma}_{\mathsf{D}}^{\dagger}\mathbf{O}_{12} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}.$$

- ⋆ This is the *Chau–Keung presentation* advocated by the PDG for both CKM and PMNS matrices.
- ⋆ Remember that the positioning of the factors in $\mathbf{V}_{(\mathsf{D})}$ is not fixed by the Murnaghan (or any other) algorithm and is just a subject-matter of agreement.
- ⋆ Today we believe we know a lot about the entries of this matrix.

Since the Dirac mass term violates conservation of the individual lepton numbers L_e, L_μ, vL_τ, it allows many lepton family number violating processes, like

$$\mu^\pm \to e^\pm + \gamma, \quad \mu^\pm \to e^\pm + e^+ + e^-,$$
$$K^+ \to \pi^+ + \mu^\pm + e^\mp, \quad K^- \to \pi^- + \mu^\pm + e^\mp,$$
$$\mu^- + (A, Z) \to e^- + (A, Z), \quad \tau^- + (A, Z) \to \mu^- + (A, Z), \ldots$$

However the $(\beta\beta)_{0\nu}$ decay or the kaon semileptonic decays like

$$K^+ \to \pi^- + \mu^+ + e^+, \quad K^- \to \pi^+ + \mu^- + e^-,$$

etc. are still forbidden as a consequence of the total lepton charge conservation.

Table 1: Current limits on the simplest lepton family number violating μ and τ decays. [From K. Nakamura *et al.*, (Particle Data Group), "Review of particle physics," J. Phys. G **37** (2010) 075021 and J. Adam *et al.* (MEG Collaboration), "New limit on the lepton-flavour violating decay $\mu^+ \to e^+\gamma$", arXiv:1107.5547 [hep-ex] (PSI-R-99-05 Experiment).].

Decay Modes	Fraction	C.L.		Decay Modes	Fraction	C.L.
$\mu^- \to e^-\nu_e\overline{\nu}_\mu$	$< 1.2\%$	90%		$\tau^- \to e^-\gamma$	$< 3.3 \times 10^{-8}$	90%
$\mu^+ \to e^+\gamma$	$< 2.4 \times 10^{-12}$	90%		$\tau^- \to \mu^-\gamma$	$< 4.4 \times 10^{-8}$	90%
$\mu^- \to e^-e^+e^-$	$< 1.0 \times 10^{-12}$	90%		$\tau^- \to e^-\pi^0$	$< 8.0 \times 10^{-8}$	90%
$\mu^- \to 2\gamma$	$< 7.2 \times 10^{-11}$	90%		$\tau^- \to \mu^-\pi^0$	$< 1.1 \times 10^{-7}$	90%

Some of the limits might seem impressive. But are they really so good?

Neutrinoless muon decay in SM

The L_μ and L_e violating muon decay $\mu^- \to e^-\gamma$ is allowed if $V^*_{\mu k} V_{ek} \neq 0$ for $k = 1, 2$ or 3. The corresponding Feynman diagrams include W loops and thus the decay width is strongly suppressed by the neutrino to W boson mass ratios:

$$R = \frac{\Gamma\left(\mu^- \to e^-\gamma\right)}{\Gamma\left(\mu^- \to e^-\nu_\mu\overline{\nu}_e\right)} = \frac{3\alpha}{32\pi}\left|\sum_k V^*_{\mu k} V_{ek} \frac{m_k^2}{m_W^2}\right|^2.$$

Since $m_k/m_W = 1.24 \times 10^{-11}\,(m_k/1\ \text{eV})$, the ratio can be estimated as

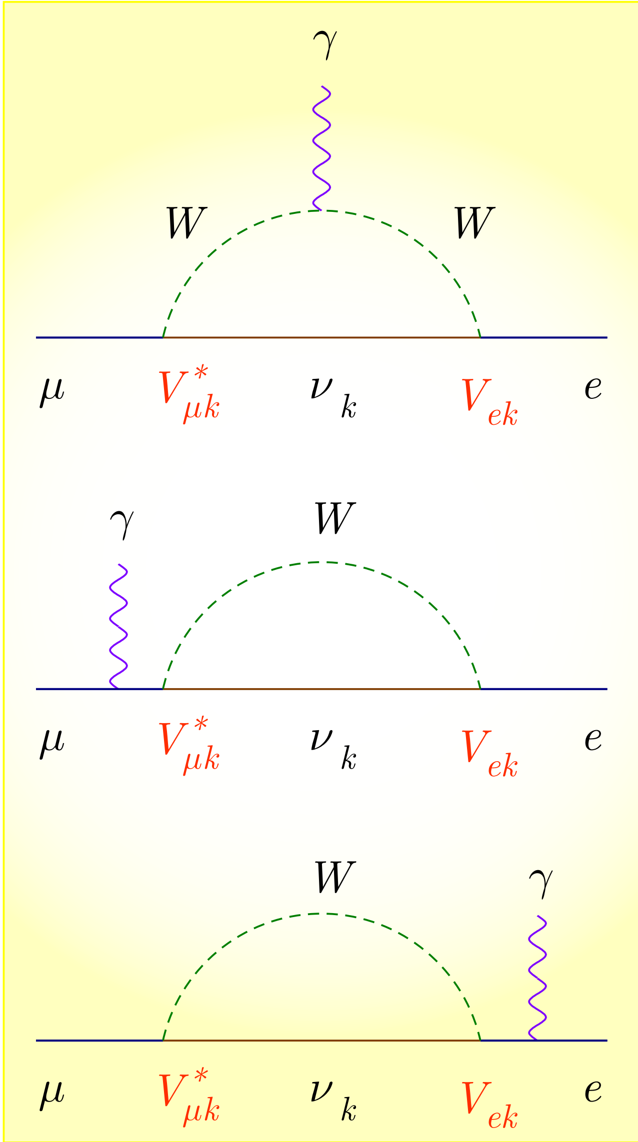
$$R \approx 5.2 \times 10^{-48}\left|\sum_k V^*_{\mu k} V_{ek}\left(\frac{m_k}{1\ \text{eV}}\right)^2\right|^2,$$

while the current experimental upper limit is (at least!) 35 orders of magnitude larger (see Table 1):

$$R_{(\text{exp})} < 2.4 \times 10^{-12} \text{ at } 90\% \text{ C.L.} \quad (\textbf{NO GO!})$$

[Some nonstandard models are bit more optimistic.]

We must deeply appreciate the oscillation phenomenon which makes the miserable ν mass effect measurable.



Majorana neutrinos

The charge conjugated bispinor field ψ^c is defined by the transformation

$$\psi \longmapsto \psi^c = C\overline{\psi}^T, \quad \overline{\psi} \longmapsto \overline{\psi^c} = -\psi^T C,$$

where C is the charge-conjugation matrix which satisfies the conditions

$$C\gamma_\alpha^T C^\dagger = -\gamma_\alpha, \quad C\gamma_5^T C^\dagger = \gamma_5, \quad C^\dagger = C^{-1} = C, \quad C^T = -C,$$

and thus coincides (up to a phase factor) with the inversion of the axes x^0 and x^2:

$$C = \gamma_0\gamma_2 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}$$

Reminder: Pauli & Dirac matrices.

$$\sigma_0 \equiv 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$$\gamma^0 = \gamma_0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}, \quad \gamma^k = -\gamma_k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \; k = 1, 2, 3, \quad \gamma^5 = \gamma_5 = -\begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix}.$$

Clearly a charged fermion field $\psi(x)$ is different from the charge-conjugated field $\psi^c(x)$.

But a neutral fermion field can coincide with the charge-conjugated field.

In other words: for a neutral fermion field $\nu(x)$ the following equality is not forbidden:

$$\boxed{\nu^c(x) = \nu(x)} \qquad \text{(Majorana condition)} \tag{2}$$

Majorana neutrino and antineutrino coincide.

"Everything which is not forbidden is allowed..." [a] $\Longrightarrow$ The real νs could be Majorana νs.

A few more details: In the chiral representation

$$\nu = \begin{pmatrix} \phi \\ \chi \end{pmatrix}, \quad \nu^c = C\overline{\nu}^T = \begin{pmatrix} -\sigma_2\chi^* \\ +\sigma_2\phi^* \end{pmatrix}.$$

$$(2) \quad \Longrightarrow \quad \phi = -\sigma_2\chi^* \quad \text{and} \quad \chi = \sigma_2\phi^* \quad \Longrightarrow \quad \phi + \chi = \sigma_2 (\phi - \chi)^*,$$

The Majorana neutrino is two-component, i.e. needs only one chiral projection. Then

$$\nu_L = \left(\frac{1+\gamma_5}{2}\right)\nu = \begin{pmatrix} \phi - \chi \\ \chi - \phi \end{pmatrix} \quad \text{and} \quad \nu_R = \left(\frac{1-\gamma_5}{2}\right)\nu = \begin{pmatrix} \phi + \chi \\ \phi + \chi \end{pmatrix} = \nu_L^c.$$

$$\Downarrow$$

$$\boxed{\nu = \nu_L + \nu_R = \nu_L + \nu_L^c.}$$

- The simplest generalization of Eq. (2), $\nu^c(x) = e^{i\varphi}\nu(x)$ ($\varphi = \text{const}$), is not very interesting.

[a] It is a constitutional principle of English law and not a law of Nature. In some other countries, it can be slightly corrected as, e.g., *"Everything is forbidden, even that which is expressly allowed."*

The Majorana mass term in the general N-neutrino case is [Gribov & Pontecorvo (1969)]:

$$\mathcal{L}_{\mathsf{M}}(x) = -\frac{1}{2}\overline{\boldsymbol{\nu}}_L^c(x)\mathbf{M}_{\mathsf{M}}\boldsymbol{\nu}_L(x) + \text{H.c.},$$

Here $\mathbf{M}_{\mathsf{M}}$ is a $N \times N$ complex *nondiagonal* matrix and, in general, $N \geq 3$.

It can be proved that the $\mathbf{M}_{\mathsf{M}}$ should be symmetric, $\mathbf{M}_{\mathsf{M}}^T = \mathbf{M}_{\mathsf{M}}$. Assuming for simplicity that its spectrum is non-degenerated, the mass matrix can be diagonalized by means of the following transformation [Bilenky & Petcov (1987)]

$$\mathbf{M}_{\mathsf{M}} = \mathbf{V}^*\mathbf{m}\mathbf{V}^\dagger, \quad \mathbf{m} = ||m_k\delta_{kl}|| = \text{diag}\,(m_1, m_2, \ldots, m_N),$$

where $\mathbf{V}$ is a unitary matrix and $m_k \geq 0$. Therefore

$$\mathcal{L}_{\mathsf{M}}(x) = -\frac{1}{2}\left[(\overline{\boldsymbol{\nu}'}_L)^c\mathbf{m}\boldsymbol{\nu}'_L + \overline{\boldsymbol{\nu}'}_L\mathbf{m}(\boldsymbol{\nu}'_L)^c\right] = -\frac{1}{2}\,\overline{\boldsymbol{\nu}'}\mathbf{m}\boldsymbol{\nu}' = -\frac{1}{2}\sum_{k=1}^{N} m_k\overline{\nu}_k\nu_k,$$

$$\boldsymbol{\nu}'_L = \mathbf{V}^\dagger\boldsymbol{\nu}_L, \quad (\boldsymbol{\nu}'_L)^c = C\left(\overline{\boldsymbol{\nu}'_L}\right)^T, \quad \boldsymbol{\nu}' = \boldsymbol{\nu}'_L + (\boldsymbol{\nu}'_L)^c.$$

The last equality means that the fields $\nu_k(x)$ are Majorana neutrino fields. Considering that the kinetic term in the neutrino Lagrangian is transformed to

$$\mathcal{L}_0 = \frac{i}{4}\,\overline{\boldsymbol{\nu}'}(x)\overleftrightarrow{\boldsymbol{\partial}}\,\boldsymbol{\nu}'(x) = \frac{i}{4}\sum_k \overline{\nu}_k(x)\overleftrightarrow{\boldsymbol{\partial}}\,\nu_k(x),$$

one can conclude that $\nu_k(x)$ is the field with the definite mass m_k.

The flavor LH neutrino fields $\nu_{\ell,L}(x)$ present in the standard weak lepton currents are linear combinations of the LH components of the fields of neutrinos with definite masses:

$$\boxed{\boldsymbol{\nu}_L = \mathbf{V}\boldsymbol{\nu}'_L \quad \text{or} \quad \nu_{\ell,L} = \sum_k V_{\ell k}\nu_{k,L}.}$$

Of course neutrino mixing matrix $\mathbf{V}$ is not the same as in the case of Dirac neutrinos.

There is no global gauge transformations under which the Majorana mass term (in its most general form) could be invariant. This implies that there are no conserved lepton charges that could allow us to distinguish Majorana νs and $\overline{\nu}$s. In other words,

Majorana neutrinos are truly neutral fermions.

Parametrization of mixing matrix for Majorana neutrinos

Since the Majorana neutrinos are not rephasable, there may be a lot of extra phase factors in the mixing matrix. The Lagrangian with the Majorana mass term is invariant with respect to the transformation

$$\ell \mapsto e^{ia_\ell}\ell, \quad V_{\ell k} \mapsto e^{-ia_\ell}V_{\ell k}$$

Therefore N phases are unphysical and the number of the physical phases now is

$$\frac{N(N+1)}{2} - N = \frac{N(N-1)}{2} = \underbrace{\frac{(N-1)(N-2)}{2}}_{\text{Dirac phases}} + \underbrace{(N-1)}_{\text{Majorana phases}} = n_{\mathsf{D}} + n_{\mathsf{M}};$$

$$n_{\mathsf{M}}(2) = 1, \quad n_{\mathsf{M}}(3) = 2, \quad n_{\mathsf{M}}(4) = 3, \ldots$$

In the case of three lepton generations one defines the diagonal matrix with the extra phase factors: $\boldsymbol{\Gamma}_{\mathsf{M}} = \mathrm{diag}\left(e^{i\alpha_1/2}, e^{i\alpha_2/2}, 1\right)$, where $\alpha_{1,2}$ are commonly referred to as the Majorana CP-violation phases. Then the PMNS matrix can be parametrized as

$$\begin{aligned}
\mathbf{V}_{(\mathsf{M})} &= \mathbf{O}_{23}\boldsymbol{\Gamma}_{\mathsf{D}}\mathbf{O}_{13}\boldsymbol{\Gamma}_{\mathsf{D}}^{\dagger}\mathbf{O}_{12}\boldsymbol{\Gamma}_{\mathsf{M}} = \mathbf{V}_{(\mathsf{D})}\boldsymbol{\Gamma}_{\mathsf{M}} \\
&= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \begin{pmatrix} e^{i\alpha_1/2} & 0 & 0 \\ 0 & e^{i\alpha_2/2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\end{aligned}$$

Neither L_ℓ nor $L = \sum_\ell L_\ell$ is conserved allowing a lot of new processes, for example, $\tau^- \to e^+(\mu^+)\pi^-\pi^-$, $\tau^- \to e^+(\mu^+)\pi^-K^-$, $\pi^- \to \mu^+\overline{\nu}_e$, $K^+ \to \pi^-\mu^+e^+$, $K^+ \to \pi^0 e^+\overline{\nu}_e$, $D^+ \to K^-\mu^+\mu^+$, $B^+ \to K^-e^+\mu^+$, $\Xi^- \to p\mu^-\mu^-$, $\Lambda_c^+ \to \Sigma^-\mu^+\mu^+$, etc.

No one was discovered yet but (may be!?) the $(\beta\beta)_{0\nu}$ decay (Heidelberg-Moscow experiment).

See-saw mechanism

Dirac-Majorana mass term for one generation

It is possible to consider mixed models in which both Majorana and Dirac mass terms are present. For simplicity sake we'll start with a **toy model** for one lepton generation.

Let us consider a theory containing two independent neutrino fields ν_L and ν_R:

$$\begin{cases} \nu_L \text{ would generally represent any active neutrino (e.g., } \nu_L = \nu_{eL}\text{)}, \\ \nu_R \text{ can represents a right handed field unrelated to any of these or} \\ \quad \text{it can be charge conjugate of any of the active neutrinos (e.g., } \nu_R = (\nu_{\mu L})^c\text{)}. \end{cases}$$

We can write the following generic mass term between ν_L and ν_R:

$$\mathcal{L}_m = -\underbrace{m_D\,\overline{\nu}_L\nu_R}_{\text{Dirac mass term}} \underbrace{-(1/2)\left[m_L\,\overline{\nu}_L\nu_L^c + m_R\,\overline{\nu}_R^c\nu_R\right]}_{\text{Majorana mass term}} + \text{H.c.} \tag{3}$$

- ⋆ As we know, the Dirac mass term respects L while the Majorana mass term violates it.
- ⋆ The parameter m_D in Eq. (3) is in general complex; to simplify matters, we'll assume it to be real but not necessarily positive.
- ⋆ The parameters m_L, and m_R in Eq. (3) can be chosen real and (by an appropriate rephasing the fields ν_L and ν_R) non-negative, but the latter is not assumed.
- ⋆ Obviously, neither ν_L nor ν_R is a mass eigenstate.

In order to obtain the *mass basis* we can apply the useful identity

$$\overline{\nu}_L \nu_R = (\overline{\nu}_R)^c (\nu_L)^c \tag{4}$$

> The identity (4) is a particular case of the more general relation
>
> $$\overline{\psi}_1 \Gamma \psi_2 = \overline{\psi}_2^c C \Gamma^T C^{-1} \psi_1^c,$$
>
> in which $\psi_{1,2}$ are Dirac spinors and Γ represents an arbitrary combination of the Dirac γ matrices.

Relation (4) allows us to rewrite Eq. (3) as follows

$$\mathcal{L}_m = -\frac{1}{2} \left(\overline{\nu}_L, (\overline{\nu}_R)^c\right) \begin{pmatrix} m_L & m_D \\ m_D & m_R \end{pmatrix} \begin{pmatrix} (\nu_L)^c \\ \nu_R \end{pmatrix} + \text{H.c.} \equiv -\frac{1}{2} \overline{\boldsymbol{\nu}}_L \mathbf{M} \left(\boldsymbol{\nu}_L\right)^c + \text{H.c.}$$

If (again for simplicity) CP conservation is assumed the matrix $\mathbf{M}$ can be diagonalized by the orthogonal transformation that is rotation

$$\mathbf{V} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \quad \text{with} \quad \theta = \frac{1}{2} \arctan\left(\frac{2m_D}{m_R - m_L}\right).$$

and we have

$$\mathbf{V}^T \mathbf{M} \mathbf{V} = \text{diag}(m_1, m_2),$$

where $m_{1,2}$ are eigenvalues of $\mathbf{M}$ given by

$$m_{1,2} = \frac{1}{2} \left(m_L + m_R \pm \sqrt{(m_L - m_R)^2 + 4m_D^2}\right).$$

Since $m_{D,L,R}$ are real, the eigenvalues are real but not necessarily positive. Let's define

$$\zeta_k = \text{sign}\, m_k$$

and rewrite the mass term in the new basis:

$$\mathcal{L}_m = -\frac{1}{2}\left[\zeta_1 |m_1| \overline{\nu}_{1L} (\nu_{1L})^c + \zeta_2 |m_2| \; (\overline{\nu}_{2R})^c \nu_{2R}\right] + \text{H.c.}, \tag{5}$$

The new fields ν_{1L} and ν_{2R} represent chiral components of two different neutrino states with "masses" m_1 and m_2, respectively:

$$\begin{pmatrix} \nu_L \\ \nu_R^c \end{pmatrix} = \mathbf{V} \begin{pmatrix} \nu_{1L} \\ \nu_{2R}^c \end{pmatrix} \quad \Longrightarrow \quad \begin{cases} \nu_{1L} = \cos\theta\, \nu_L - \sin\theta\, \nu_R^c, \\ \nu_{2R} = \sin\theta\, \nu_L^c + \cos\theta\, \nu_R. \end{cases}$$

Now we define two 4-component fields

$$\nu_1 = \nu_{1L} + \zeta_1 (\nu_{1L})^c \quad \text{and} \quad \nu_2 = \nu_{2R} + \zeta_2 (\nu_{2R})^c.$$

Certainly, these fields are self-conjugate with respect to the C transformation:

$$\nu_k^c = \zeta_k \nu_k \quad (k = 1, 2)$$

and therefore they describe Majorana neutrinos. In terms of these fields Eq. (5) reads

$$\mathcal{L}_m = -\frac{1}{2}\left(|m_1| \overline{\nu}_1 \nu_1 + |m_2| \overline{\nu}_2 \nu_2\right). \tag{6}$$

We therefore conclude that $\nu_k(x)$ is the Majorana neutrino field with the definite (physical) mass $|m_k|$.

There are several special cases of the Dirac-Majorana mass matrix $\mathbf{M}$ which are of considerable phenomenological importance, in particular,

$$\text{(A):}\quad \mathbf{M} = \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix} \quad \Longrightarrow \quad |m_{1,2}| = m, \quad \theta = \frac{\pi}{4} \quad \text{(maximal mixing).}$$

Two Majorana fields are equivalent to one Dirac field.

A generalization $|m_{L,R}| \ll |m_D|$, leads to the so-called Pseudo-Dirac neutrinos.

$$\text{(B):}\quad \mathbf{M} = \begin{pmatrix} m_L & m \\ m & m_L \end{pmatrix} \quad \Longrightarrow \quad m_{1,2} = m_L \pm m_D, \quad \theta = \frac{\pi}{4} \quad \text{(maximal mixing);}$$

$$\text{(C):}\quad \mathbf{M} = \begin{pmatrix} 0 & m \\ m & M \end{pmatrix} \quad \text{or, more generally,} \quad |m_L| \ll |m_R|, \quad m_D > 0.$$

The see-saw

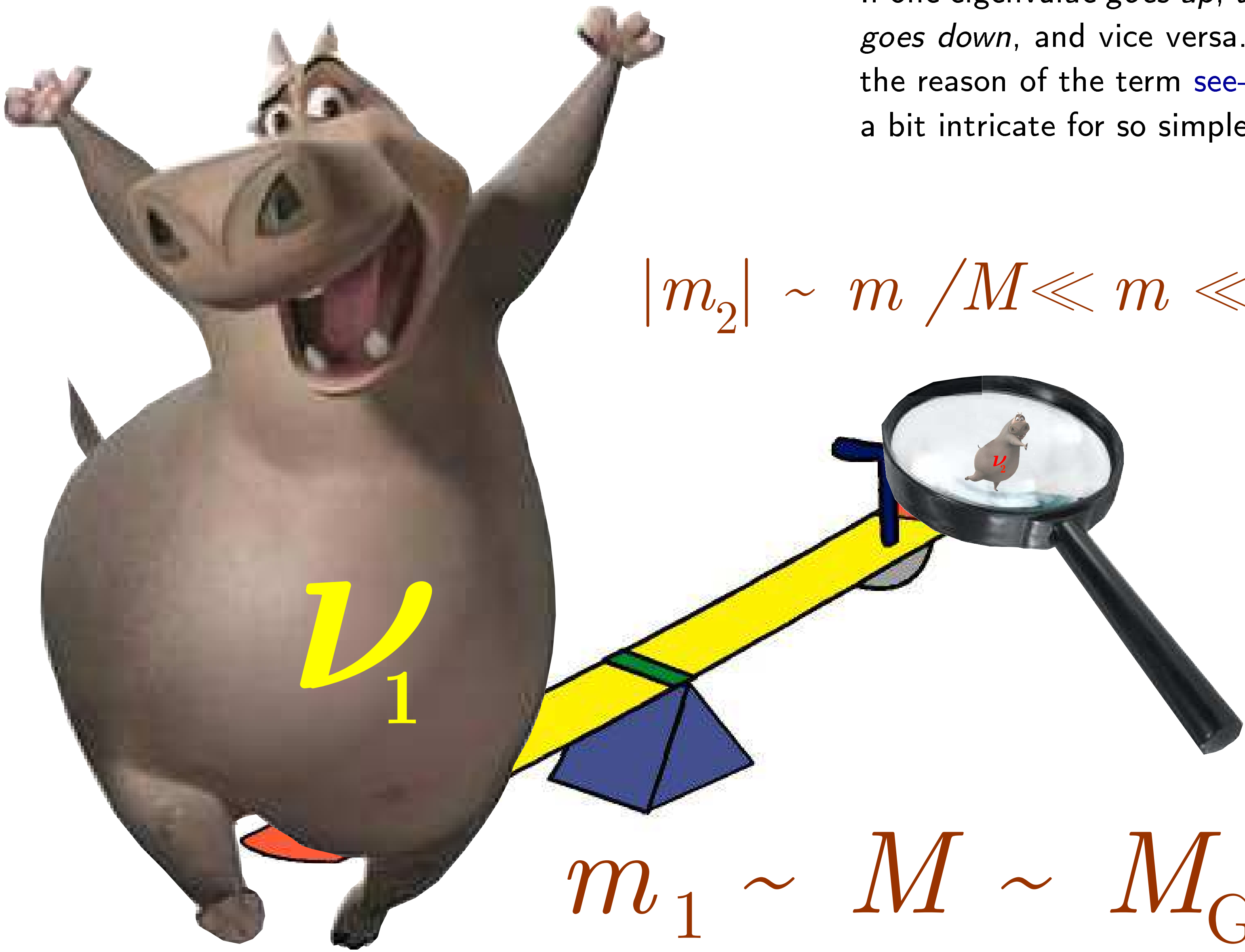
The case (C) with $m \ll M$ is the simplest example of the see-saw mechanism. It leads to two masses, one very large, $m_1 \approx M$, other very small, $m_2 \approx -m^2/M \ll m$, suppressed compared to the entries in $\mathbf{M}$. In particular, one can assume

$$m \sim m_\ell \text{ or } m_q \quad (0.5\text{ MeV to } 200\text{ GeV}) \quad \text{and} \quad M \sim M_{\mathsf{GUT}} \sim 10^{15-16}\text{ GeV}.$$

Then $|m_2|$ can ranges from $\sim 10^{-14}$ eV to ~ 0.04 eV. The mixing between the heavy and light neutrinos is extremely small: $\theta \approx m/M \sim 10^{-20} - 10^{-13} \lll 1$.

If one eigenvalue *goes up*, the other *goes down*, and vice versa. This is the reason of the term see-saw... a bit intricate for so simple idea...

$$|m_2| \sim m\ /M \ll m \ll M$$



$$m_1 \sim M \sim M_{\mathrm{GUT}}$$

More neutral fermions

A generalization of the above scheme to N generations is almost straightforward but technically rather cumbersome. Let's consider it schematically for the $N = 3$ case.

- ▷ If neutral fermions are added to the set of the SM fields, then the flavour neutrinos can acquire mass by mixing with them.
- ▷ The additional fermions can be[a]
  - Gauge chiral singlets per family $\mathcal{N}$ (e.g., right-handed neutrinos) [Type I seesaw], or
  - $SU(2) \times U(1)$ doublets (e.g., Higgsino in SUSY), or
  - $Y = 0$, $SU(2)_L$ triplets Σ (e.g., Wino in SUSY) [Type III seesaw].
- ▷ Addition of three right-handed neutrinos $\mathcal{N}_{iR}$ leads to the see-saw mechanism with the following mass terms:

$$\mathcal{L}_m = -\sum_{ij} \left[\overline{\nu}_{iL} M^D_{ij} \mathcal{N}_{jR} - \frac{1}{2} \left(\mathcal{N}_{iR}\right)^c M^R_{ij} \mathcal{N}_{jR} + \text{H.c.} \right].$$

- ▷ The above equation leads to the following 6×6 see-saw mass matrix:

$$\mathbf{M} = \begin{pmatrix} \mathbf{0} & \mathbf{m}_D^T \\ \mathbf{m}_D & \mathbf{M}_R \end{pmatrix}.$$

  Both $\mathbf{m}_D$ and $\mathbf{m}_R$ are 3×3 matrices in the generation space.

[a] Type II seesaw operates with additional $SU(2)_L$ scalar triplets Δ.

Similar to the one-generation case we assume that the eigenvalues of $\mathbf{m}_R$ are large in comparison with the eigenvalues of $\mathbf{m}_D$. Then $\mathbf{M}$ can be approximately block-diagonalized by an unitary transformation:

$$\mathbf{U}^\dagger \mathbf{M} \mathbf{U} = \mathrm{diag}\left(\mathbf{M}_1, \mathbf{M}_2\right) + \mathcal{O}\left(\mathbf{m}_D \mathbf{M}_R^{-1}\right),$$

where

$$\mathbf{U} = \begin{pmatrix} 1 + \frac{1}{2}\mathbf{m}_D^\dagger \left(\mathbf{M}_R \mathbf{M}_R^\dagger\right)^{-1} \mathbf{m}_D & \mathbf{m}_D^\dagger \left(\mathbf{M}_R^\dagger\right)^{-1} \\ -\mathbf{M}_R^{-1}\mathbf{m}_D & 1 + \frac{1}{2}\mathbf{M}_R^{-1}\mathbf{m}_D \mathbf{m}_D^\dagger \left(\mathbf{M}_R^\dagger\right)^{-1} \end{pmatrix}.$$

$$\mathbf{M}_1 \simeq -\mathbf{m}_D^T \mathbf{M}_R^{-1} \mathbf{m}_D \quad \text{and} \quad \mathbf{M}_2 \simeq \mathbf{M}_R.$$

The mass eigenfields are surely Majorana neutrinos.

- Quadratic see-saw: If eigenvalues of $\mathbf{M}_R$ are of the order of a large scale parameter $M \sim M_{\mathsf{GUT}}$[a] [e.g., $\mathbf{M}_R = M\mathbf{1}$] than the standard neutrino masses are suppressed:

$$m_i \sim \frac{m_{Di}^2}{M} \lll m_{Di},$$

Here $m_{Di} \sim Y_i \langle H \rangle$ are the eigenvalues of $\mathbf{m}_D$. As long as these eigenvalues (or Yukawa couplings Y_i) are hierarchical, the Majorana neutrino masses display quadratic hierarchy:

$$m_1 : m_2 : m_3 \propto m_{D1}^2 : m_{D2}^2 : m_{D3}^2.$$

[a] Large M is natural in, e.g., $SO(10)$ inspired GUT theories which therefore provide a nice framework to understand small neutrino masses [see, e.g., poster presentation by Rohit Verma *et al.* in this School.]

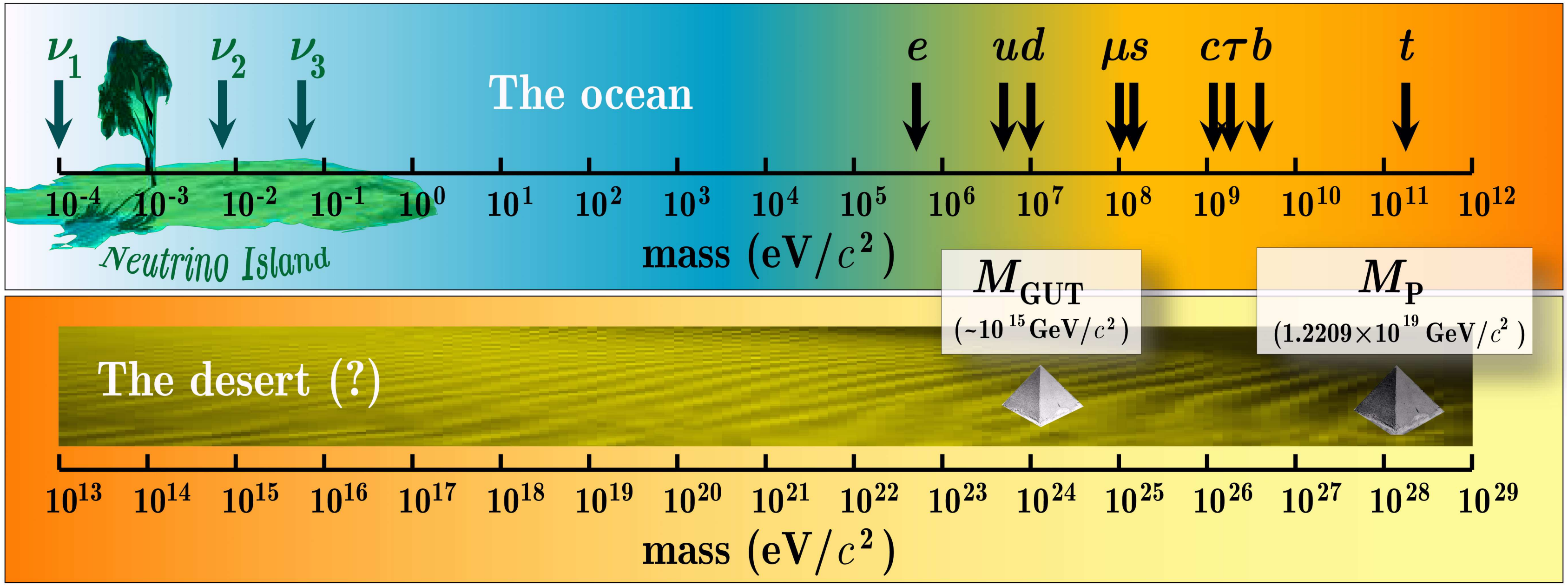
- Linear see-saw: In a more special case, $\mathbf{M}_R = (M/M_D)\mathbf{M}_D$, where M_D is the generic scale of the charged fermion masses than

$$m_i \sim \frac{M_D m_{Di}}{M} \lll m_{Di}$$

but the hierarchy is linear:

$$m_1 : m_2 : m_3 \propto m_{D1} : m_{D2} : m_{D3}.$$

The two mentioned possibilities are, in principle, experimentally distinguishable.



Double see-saw & inverse see-saw

The see-saw can be implemented by introducing additional neutrino singlets beyond the three RH neutrinos involved into the see-saw type I. One have to distinguish between

- RH neutrinos $\boldsymbol{\nu}_R$, which carry $B-L$ and perhaps (not necessary) form $SU(2)_R$ doublets with RH charged leptons, and
- Neutrino singlets $\boldsymbol{\nu}_S$, which have no Yukawa couplings to the LH neutrinos but may couple to $\boldsymbol{\nu}_R$.

If the singlets have nonzero Majorana masses $\mathbf{M}_{SS}$ while the RH neutrinos have a zero Majorana mass, $\mathbf{M}_{RR}=0$, the see-saw mechanism may proceed via mass couplings of the singlets to RH neutrinos, $\mathbf{M}_{RS}$. In the basis $(\boldsymbol{\nu}_L,\boldsymbol{\nu}_R,\boldsymbol{\nu}_S)$, the 9×9 mass matrix is

$$\begin{pmatrix} \mathbf{0} & \mathbf{m}_{LR} & \mathbf{0} \\ \mathbf{m}_{LR} & \mathbf{0} & \mathbf{M}_{RS} \\ \mathbf{0} & \mathbf{M}_{RS}^T & \mathbf{M}_{SS} \end{pmatrix}.$$

Assuming that the eigenvalues of $\mathbf{M}_{SS}$ are much smaller than the eigenvalues of $\mathbf{M}_{RS}$, the light physical LH Majorana neutrino masses are then doubly suppressed,

$$\mathbf{M}_1 \simeq \mathbf{m}_{LR}\mathbf{M}_{RS}^{-1}\mathbf{M}_{SS}\left(\mathbf{M}_{RS}^T\right)^{-1}\mathbf{m}_{LR}^T, \quad \mathbf{M}_2^2 \simeq \mathbf{M}_{RS}^2+\mathbf{m}_{LR}^2.$$

This scenario is usually used in string inspired models [see, e.g., R.N.Mohapatra & J.W.Valle, Phys. Rev. D **34** (1986) 1642; M.C.Gonzalez-Garcia & J.W.F.Valle, Phys. Lett. B **216** (1989) 360].

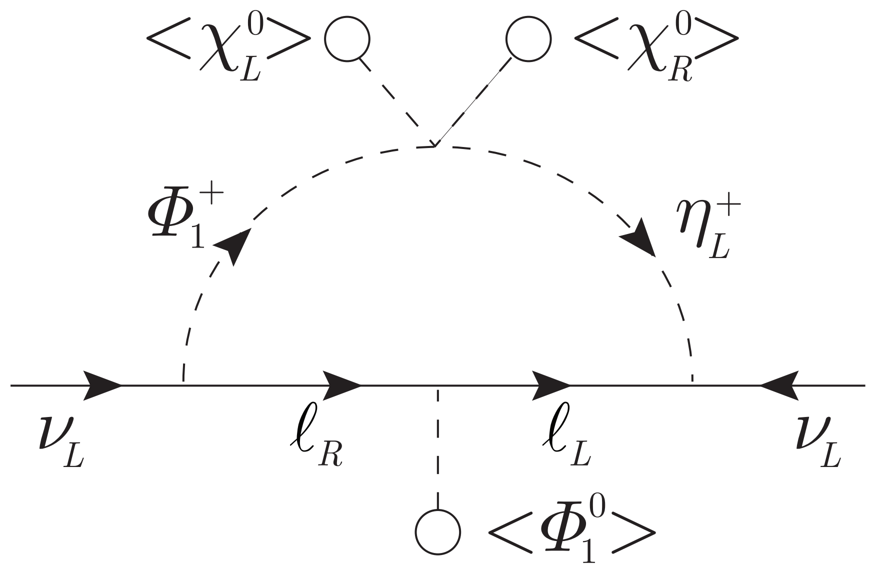
Radiative see-saw

An alternative mechanism relies on the radiative generation of neutrino masses [H.Georgi & S.L.Glashow, Phys. Rev. D **7** (1973) 2487; P.Cheng & L.-F.Li, Phys. Rev. D **17** (1978) 2375; Phys. Rev. D **22** (1980) 2860; A.Zee, Phys. Lett. B **93** (1980) 389;. . ..] In this scheme, the neutrinos are massless at the tree level, but pick up small masses due to loop corrections.

In a typical model [K.S. Babu & V.S. Mathur, Phys. Rev. D **11** (1988) 3550] the see-saw formula is modified as

$$m_\nu \sim \left(\frac{\alpha}{\pi}\right)\frac{m_l^2}{M},$$

where the prefactor $\alpha/\pi \approx 2\times 10^{-3}$ arises due to the loop structure of the neutrino mass diagram. Light neutrinos are now possible even for relatively "light" mass scale M of "new physics." The scalar sector consists of the multiplets



$$\chi_{L,R} = (\chi^+,\chi^0)_{L,R}, \quad \Phi = \begin{pmatrix} \Phi_1^0 & \Phi_2^+ \\ \Phi_1^- & \Phi_2^0 \end{pmatrix}, \quad \eta^+_{L,R}.$$

The diagram in the figures is responsible for generation of Majorana masses for ν_L. The analogous diagram is obtained by the replacement $L \to R$ and $\Phi_1^+ \to \Phi_2^+$.

Beyond this lecture

- SUSY & SUGRA see-saw
- TeV see-saw & Large Extra Dimensions
- Dirac see-saw
- Top (top-bottom) see-saw
- See-saw & Dark Matter
- See-saw & Leptogenesis
- ...

Conclusions [are not actually validated]

- The "mainstream" neutrino mass models, defined as see-saw models, are capable of describing the atmospheric–reactor–accelerator neutrino oscillation data, the LMA MSW solar neutrino solution, and cosmological limits.
- The Standard Model and the Minimal Supersymmetric Standard Model may naturally be extended to incorporate the see-saw mechanism.
- [A fly in the ointment] Wealth of the models ($\gg$ number of the authors of the models) greatly complicates the choice of the best one.

BACKUP SLIDES

Bi-unitary diagonalization

Let's prove that any *nonsingular* matrix $\mathbf{M}$ can be diagonalized by a bi-unitary transformation.

Proof. Since $\mathbf{M}\mathbf{M}^\dagger$ is Hermitian, there exist a unitary matrix $\mathbf{V}$ such that

$$\mathbf{V}^\dagger \left(\mathbf{M}\mathbf{M}^\dagger\right) \mathbf{V} = \mathbf{m}^2 = \mathsf{diag}\left(m_1^2, m_2^2, \ldots, m_N^2\right),$$

where m_i^2 are real for any i. Moreover $m_i^2 > 0$. Indeed, $\mathbf{M}^\dagger\mathbf{V} = \left(\mathbf{V}^\dagger\mathbf{M}\right)^\dagger$ and thus

$$m_i^2 = \sum_j \left(\mathbf{V}^\dagger\mathbf{M}\right)_{ij} \left(\mathbf{V}^\dagger\mathbf{M}\right)^*_{ij} = \sum_j \left|\left(\mathbf{V}^\dagger\mathbf{M}\right)_{ij}\right|^2 \geq 0;$$

the equality is however excluded since $\mathbf{m}^2$ is nonsingular. Let's now define the matrix

$$\widetilde{\mathbf{V}} = \mathbf{M}^\dagger\mathbf{V}\mathbf{m}^{-1}.$$

$$\Downarrow$$

$$\widetilde{\mathbf{V}}^\dagger = \mathbf{m}^{-1}\mathbf{V}^\dagger\mathbf{M}$$

$$\Downarrow$$

$$\widetilde{\mathbf{V}}^\dagger\widetilde{\mathbf{V}} = \mathbf{m}^{-1}\mathbf{V}^\dagger\mathbf{M}\mathbf{M}^\dagger\mathbf{V}\mathbf{m}^{-1} = \mathbf{m}^{-1}\mathbf{m}^2\mathbf{m}^{-1} = \mathbf{1},$$

that is $\widetilde{\mathbf{V}}$ is unitary and

$$\mathbf{V}^\dagger\mathbf{M}\widetilde{\mathbf{V}} = \mathbf{m}.$$

Q.E.D.

TeV-scale gauged $B-L$ symmetry with Inverse see-saw

Consider briefly one more inverse see-saw model [S.Khalil, Phys. Rev. D **82** (2010) 077702].

The model is based on the following:

(i) The SM singlet Higgs boson, which breaks the $B-L$ gauge symmetry, has $B-L$ unit charge.

(ii) The SM singlet fermion sector includes two singlet fermions $S_{\pm}$ with $B-L$ charges ± 2 with opposite matter parity.

The Lagrangian of neutrino masses, in the flavor basis, is given by

$$\overline{\boldsymbol{\nu}}_L \mathbf{m}_D \boldsymbol{\nu}_R + \boldsymbol{\nu}_R^c \mathbf{M}_N S_- + \mu_s \overline{\boldsymbol{S}}_- \boldsymbol{S}_- .$$

In the limit $\mu_s \to 0$, which corresponds to the unbroken $(-1)^{L+S}$ symmetry, the light neutrinos remain massless. Therefore, a small nonvanishing μ_s can be considered as a slight breaking of a this global symmetry and the smallness of μ_s is natural. Small μ_s can also be generated radiatively.

In the basis $(\boldsymbol{\nu}_L, \boldsymbol{\nu}_R^c, \boldsymbol{S}_-)$, the 9×9 mass matrix is

$$\begin{pmatrix} \mathbf{0} & \mathbf{m}_D & \mathbf{0} \\ \mathbf{m}_D^T & \mathbf{0} & \mathbf{M}_N \\ \mathbf{0} & \mathbf{M}_N^T & \mu_s \end{pmatrix}.$$

So, up to the notation, it reproduces all the properties of the double see-saw.